SS3 MATHEMATICS LESSON NOTES

FIRST TERM

WEEK 1: REVIEW OF LOGARITHMS

CONTENT

1. Definition and Basic Concepts

A **logarithm** is the inverse operation of exponentiation. If $\mathbf{a^x = y}$, then $\mathbf{\log\_a(y) = x}$

Where:

- **a** = base of the logarithm ($a > 0$, $a \neq 1$)
- **x** = the logarithm (the power/exponent)
- **y** = the number ($y > 0$)

Read as: "log to base a of y equals x"

Relationship:

- **Exponential form:** $a^x = y$
- **Logarithmic form:** $\log\_a(y) = x$

Examples:

1. $\mathbf{2^3 = 8}$ can be written as $\mathbf{\log_2(8) = 3}$
2. $\mathbf{10^2 = 100}$ can be written as $\mathbf{\log_{10}(100) = 2}$
3. $\mathbf{5^2 = 25}$ can be written as $\mathbf{\log_5(25) = 2}$
4. $\mathbf{3^4 = 81}$ can be written as $\mathbf{\log_3(81) = 4}$

2. Laws of Logarithms

Law 1: Product Law (Multiplication) $\mathbf{\log\_a(m \times n) = \log\_a(m) + \log\_a(n)}$

The logarithm of a product equals the sum of the logarithms.

Example 1: $\log_2(8 \times 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$

Verify: $8 \times 4 = 32$, and $2^5 = 32$ ✓

Example 2: $\log_{10}(100 \times 1000) = \log_{10}(100) + \log_{10}(1000) = 2 + 3 = 5$

Verify: $100 \times 1000 = 100{,}000 = 10^5$ ✓

Law 2: Quotient Law (Division) log_a(m ÷ n) = log_a(m) - log_a(n)

The logarithm of a quotient equals the difference of the logarithms.

Example 1: $\log_2(16 \div 4) = \log_2(16) - \log_2(4) = 4 - 2 = 2$

Verify: $16 \div 4 = 4$, and $2^2 = 4$ ✓

Example 2: $\log_5(125 \div 25) = \log_5(125) - \log_5(25) = 3 - 2 = 1$

Verify: $125 \div 25 = 5$, and $5^1 = 5$ ✓

Law 3: Power Law log_a(m^n) = n × log_a(m)

The logarithm of a number raised to a power equals the power times the logarithm of the number.

Example 1: $\log_2(8^3) = 3 \times \log_2(8) = 3 \times 3 = 9$

Verify: $8^3 = 512$, and $2^9 = 512$ ✓

Example 2: $\log_{10}(100^2) = 2 \times \log_{10}(100) = 2 \times 2 = 4$

Verify: $100^2 = 10{,}000 = 10^4$ ✓

Law 4: Logarithm of 1 log_a(1) = 0 for any base a

Because $a^0 = 1$ for any $a \neq 0$

Examples:

- $\log_2(1) = 0$
- $\log_{10}(1) = 0$
- $\log_5(1) = 0$

Law 5: Logarithm of the Base log_a(a) = 1 for any base a

Because $a^1 = a$

Examples:

- $\log_2(2) = 1$
- $\log_{10}(10) = 1$
- $\log_5(5) = 1$

Law 6: Change of Base Formula log_a(m) = log_b(m) ÷ log_b(a)

This allows us to convert logarithms from one base to another.

Example: Convert $\log_2(8)$ to base 10:

$\log_2(8) = \log_{10}(8) \div \log_{10}(2) = 0.9031 \div 0.3010 = 3$ ✓

3. Solving Logarithmic Equations

Type 1: Simple Logarithmic Equations

Example 1: Solve: $\log_2(x) = 5$

Solution: Convert to exponential form: $x = 2^5$ $x = 32$

Example 2: Solve: $\log_{10}(x) = 3$

Solution: $x = 10^3$ $x = 1000$

Example 3: Solve: $\log_5(x) = -2$

Solution: $x = 5^{-2}$ $x = 1/25$ $x = 0.04$

Type 2: Logarithmic Equations Using Laws

Example 4: Solve: $\log_2(x) + \log_2(x - 3) = 2$

Solution: Step 1: Apply product law $\log_2[x(x - 3)] = 2$

Step 2: Convert to exponential form $x(x - 3) = 2^2$ $x^2 - 3x = 4$

Step 3: Rearrange $x^2 - 3x - 4 = 0$

Step 4: Factorize $(x - 4)(x + 1) = 0$ $x = 4$ or $x = -1$

Step 5: Check validity (logarithm requires positive numbers) For $x = 4$: $\log_2(4) + \log_2(1) = 2 + 0 = 2$ ✓ For $x = -1$: Invalid (cannot take log of negative number)

Answer: x = 4

Example 5: Solve: $\log_3(x + 2) - \log_3(x - 1) = 1$

Solution: Step 1: Apply quotient law $\log_3[(x + 2)/(x - 1)] = 1$

Step 2: Convert to exponential form $(x + 2)/(x - 1) = 3^1$ $(x + 2)/(x - 1) = 3$

Step 3: Cross multiply $x + 2 = 3(x - 1)$ $x + 2 = 3x - 3$

Step 4: Solve $2 + 3 = 3x - x$ $5 = 2x$ $x = 5/2 = 2.5$

Step 5: Verify $\log_3(2.5 + 2) - \log_3(2.5 - 1) = \log_3(4.5) - \log_3(1.5) = \log_3(3) = 1$ ✓

Answer: x = 2.5

Example 6: Solve: $2\log_5(x) = \log_5(9)$

Solution: Step 1: Apply power law (reverse) $\log_5(x^2) = \log_5(9)$

Step 2: Since bases and logs are equal $x^2 = 9$

Step 3: Solve $x = \pm 3$

Step 4: Check validity Only $x = 3$ is valid (x must be positive)

Answer: $x = 3$

Type 3: Exponential Equations Requiring Logarithms

Example 7: Solve: $2^x = 10$

Solution: Step 1: Take log of both sides $\log_{10}(2^x) = \log_{10}(10)$

Step 2: Apply power law $x \times \log_{10}(2) = 1$

Step 3: Solve $x = 1 \div \log_{10}(2)$ $x = 1 \div 0.3010$ $x = 3.322$

Answer: $x \approx 3.32$

Example 8: Solve: $3^{2x+1} = 27$

Solution: Method 1: Using indices $3^{2x+1} = 3^3$ $2x + 1 = 3$ $2x = 2$ $x = 1$

Method 2: Using logarithms $\log_3(3^{2x+1}) = \log_3(27)$ $(2x + 1)\log_3(3) = \log_3(27)$ $2x + 1 = 3$ $x = 1$

Answer: $x = 1$

4. Application of Logarithms to Complex Calculations

Logarithms simplify complex calculations involving multiplication, division, powers, and roots.

Example 9: Multiplication using logarithms Calculate: 47.5×32.8

Solution: Let $y = 47.5 \times 32.8$

Step 1: Take log of both sides $\log_{10}(y) = \log_{10}(47.5 \times 32.8)$

Step 2: Apply product law $\log_{10}(y) = \log_{10}(47.5) + \log_{10}(32.8)$

Step 3: Use log tables $\log_{10}(47.5) = 1.6767$ $\log_{10}(32.8) = 1.5159$

Step 4: Add $\log_{10}(y) = 1.6767 + 1.5159 = 3.1926$

Step 5: Find antilog $y = \text{antilog}(3.1926)$ $y = 1558$

Answer: 1558

Example 10: Division using logarithms Calculate: $875 \div 23.4$

Solution: Let $y = 875 \div 23.4$

Step 1: Take log $\log_{10}(y) = \log_{10}(875) - \log_{10}(23.4)$

Step 2: Use log tables $\log_{10}(875) = 2.9420$ $\log_{10}(23.4) = 1.3692$

Step 3: Subtract $\log_{10}(y) = 2.9420 - 1.3692 = 1.5728$

Step 4: Find antilog $y = \text{antilog}(1.5728)$ $y = 37.4$

Answer: 37.4

Example 11: Power using logarithms Calculate: $(4.8)^4$

Solution: Let $y = (4.8)^4$

Step 1: Take log $\log_{10}(y) = \log_{10}(4.8^4)$

Step 2: Apply power law $\log_{10}(y) = 4 \times \log_{10}(4.8)$

Step 3: Use log table $\log_{10}(4.8) = 0.6812$

Step 4: Multiply $\log_{10}(y) = 4 \times 0.6812 = 2.7248$

Step 5: Find antilog $y = \text{antilog}(2.7248)$ $y = 530.8$

Answer: 530.8

Example 12: Root using logarithms Calculate: $\sqrt[4]{625}$

Solution: Let $y = \sqrt[4]{625} = 625^{(1/4)}$

Step 1: Take log $\log_{10}(y) = \log_{10}(625^{(1/4)})$

Step 2: Apply power law $\log_{10}(y) = (1/4) \times \log_{10}(625)$

Step 3: Use log table $\log_{10}(625) = 2.7959$

Step 4: Multiply $\log_{10}(y) = (1/4) \times 2.7959 = 0.6990$

Step 5: Find antilog $y = \text{antilog}(0.6990)$ $y = 5$

Answer: 5

Example 13: Combined operations Calculate: $(67.2 \times 8.5) \div \sqrt{38.4}$

Solution: Let $y = (67.2 \times 8.5) \div \sqrt{38.4}$

Step 1: Take log $\log_{10}(y) = \log_{10}(67.2) + \log_{10}(8.5) - \log_{10}(38.4^{(1/2)})$ $\log_{10}(y) = \log_{10}(67.2) + \log_{10}(8.5) - (1/2)\log_{10}(38.4)$

Step 2: Use log tables $\log_{10}(67.2) = 1.8274$ $\log_{10}(8.5) = 0.9294$ $\log_{10}(38.4) = 1.5843$

Step 3: Calculate $\log_{10}(y) = 1.8274 + 0.9294 - (0.5 \times 1.5843)$ $\log_{10}(y) = 2.7568 - 0.7922$ $\log_{10}(y) = 1.9646$

Step 4: Find antilog $y = \text{antilog}(1.9646)$ $y = 92.2$

Answer: 92.2

5. Common Logarithm (Base 10) and Natural Logarithm (Base e)

Common Logarithm (log or $\log_{10}$):

- Base 10
- Used in scientific calculations
- Available on calculators as "log" button

Natural Logarithm (ln or log_e):

- Base e ($e \approx 2.71828...$)
- Used in calculus, growth/decay problems
- Available on calculators as "ln" button

Relationship: $\ln(x) = \log_e(x)$ $\log(x) = \log_{10}(x)$

Conversion: $\ln(x) = 2.3026 \times \log_{10}(x)$

EVALUATION

1. Express in logarithmic form: (a) $5^3 = 125$ (b) $10^4 = 10{,}000$ (c) $2^{-3} = 1/8$
2. Evaluate without using tables: (a) $\log_5(25)$ (b) $\log_3(81)$ (c) $\log_2(1/16)$
3. Simplify: $\log_{10}(1000) - \log_{10}(10) + \log_{10}(100)$
4. Solve: (a) $\log_2(x) = 4$ (b) $\log_5(x) = -1$ (c) $\log_{10}(x) = 2.5$
5. Given that $\log_{10}(2) = 0.3010$ and $\log_{10}(3) = 0.4771$, find $\log_{10}(6)$
6. Solve: $\log_3(x) + \log_3(x + 6) = 3$
7. Simplify: $2\log_2(4) + 3\log_2(2) - \log_2(8)$
8. Solve: $\log_{10}(x + 3) - \log_{10}(x - 2) = 1$

9. Use logarithms to evaluate: 45.6   78.2

10. Solve: $5^x = 20$ (Use $\log_{10}(2) = 0.3010$, $\log_{10}(5) = 0.6990$)

ASSIGNMENT

1. **Simplification Problems:** a) Simplify: $\log_5(125) + \log_5(25) - \log_5(5)$ b) Simplify: $3\log_2(8) - 2\log_2(4) + \log_2(16)$ c) Express as a single logarithm: $2\log_{10}(x) + \log_{10}(y) - 3\log_{10}(z)$

2. **Equation Solving:** a) Solve: $\log_3(2x - 1) = 2$ b) Solve: $\log_2(x) + \log_2(x - 7) = 3$ c) Solve: $2\log_5(x) - \log_5(x - 4) = \log_5(4)$ d) Solve: $3^{x+2} = 81$

3. **Application Problems:** a) Given $\log_{10}(2) = 0.3010$, $\log_{10}(3) = 0.4771$, and $\log_{10}(7) = 0.8451$, evaluate: i) $\log_{10}(14)$ ii) $\log_{10}(21)$ iii) $\log_{10}(4.5)$

   b) Use logarithms to calculate: i) 87.3   42.6 ii) 456 ÷ 12.8 iii) $(3.6)^3$ iv) $\sqrt[3]{216}$

4. **Word Problem:** The population of a city grows exponentially according to the formula $P = P_0$ 2^(t/10), where P_0 is the initial population and t is time in years. a) If the initial population is 50,000, how long will it take for the population to reach 200,000? b) Use logarithms to solve your equation.

WEEK 2: LOGIC

CONTENT

1. Introduction to Logic

Logic is the science of reasoning. It deals with methods of distinguishing correct reasoning from incorrect reasoning. In mathematics, logic provides a systematic way to determine the validity of statements and arguments.

Statement: A statement (or proposition) is a declarative sentence that is either **true** or **false**, but not both.

Examples of Statements:

- "Lagos is in Nigeria" (True)
- "5 + 3 = 7" (False)
- "All rectangles have four sides" (True)

Examples of Non-Statements:

- "Go to school!" (Command - not true or false)
- "What is your name?" (Question - not true or false)
- "x + 5 = 10" (Neither true nor false until x is specified)

2. Simple and Compound Statements

Simple Statement: A statement that contains only one idea and cannot be broken down into simpler statements.

Examples:

- p: "It is raining"
- q: "I am hungry"
- r: "2 is an even number"

Compound Statement: A statement formed by combining two or more simple statements using logical connectives.

Examples:

- "It is raining **and** I am hungry"
- "2 is even **or** 2 is odd"
- "**If** you study hard, **then** you will pass"

3. Logical Operations and Their Symbols

A. Negation (NOT) - Symbol: ~, ¬, or '

The negation of a statement is the opposite of the statement.

Symbol: ~p (read as "not p")

Example:

- p: "It is raining"
- ~p: "It is not raining"

Truth Value:

- If p is true, then ~p is false
- If p is false, then ~p is true

Truth Table for Negation:

p	~p
T	F
F	T

B. Conjunction (AND) - Symbol: ∧

A conjunction connects two statements with "and." It is true only when **both** statements are true.

Symbol: p ∧ q (read as "p and q")

Example:

- p: "It is sunny"
- q: "It is warm"
- p ∧ q: "It is sunny **and** it is warm"

Truth Table for Conjunction:

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Key Point: Both must be true for conjunction to be true.

C. Disjunction (OR) - Symbol: ∨

A disjunction connects two statements with "or." It is true when **at least one** statement is true.

Symbol: p ∨ q (read as "p or q")

Example:

- p: "I will study Mathematics"
- q: "I will study English"
- p ∨ q: "I will study Mathematics **or** English"

Truth Table for Disjunction:

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Key Point: At least one must be true for disjunction to be true.

D. Conditional (If...Then) - Symbol: →

A conditional statement expresses "if p, then q."

Symbol: p → q (read as "if p, then q" or "p implies q")

Parts:

- **p** = hypothesis (antecedent)
- **q** = conclusion (consequent)

Example:

- p: "It rains"
- q: "The ground is wet"
- p → q: "**If** it rains, **then** the ground is wet"

Truth Table for Conditional:

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Key Point: The conditional is false only when the hypothesis is true but the conclusion is false.

Understanding the Conditional:

- If the hypothesis is false, the conditional is automatically true (vacuous truth)
- A false hypothesis doesn't tell us anything about the conclusion

E. Biconditional (If and Only If) - Symbol: ↔

A biconditional states that p and q are equivalent; both must have the same truth value.

Symbol: $p \leftrightarrow q$ (read as "p if and only if q")

Example:

- p: "A number is divisible by 2"
- q: "A number is even"
- $p \leftrightarrow q$: "A number is divisible by 2 **if and only if** it is even"

Truth Table for Biconditional:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Key Point: True when both statements have the same truth value (both true or both false).

4. Constructing Truth Tables

Example 1: Truth table for $\sim(p \land q)$

Step-by-step:

p	q	$p \land q$	$\sim(p \land q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

Explanation:

1. List all possible combinations of p and q
2. Calculate $p \land q$
3. Negate the result

Example 2: Truth table for $(p \lor q) \land \sim p$

p	q	$p \lor q$	$\sim p$	$(p \lor q) \land \sim p$
T	T	T	F	F
T	F	T	F	F
F	T	T	T	T
F	F	F	T	F

Explanation:

1. Calculate p ∨ q
2. Calculate ~p
3. Find conjunction of results

Example 3: Truth table for (p → q) ∧ (q → p)

p	q	p → q	q → p	(p → q) ∧ (q → p)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note: This is equivalent to p ↔ q (biconditional)

Example 4: Complex statement - (p ∧ q) → r

p	q	r	p ∧ q	(p ∧ q) → r
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

5. Conditional Statements and Their Variations

For a conditional statement **p → q**, there are three related statements:

A. Converse: q → p

- Switch hypothesis and conclusion
- Example: "If the ground is wet, then it rained"

B. Inverse: ~p → ~q

- Negate both hypothesis and conclusion
- Example: "If it doesn't rain, then the ground is not wet"

C. Contrapositive: ~q → ~p

- Switch AND negate both parts
- Example: "If the ground is not wet, then it didn't rain"

Important Relationship:

- A conditional and its **contrapositive** are **logically equivalent** (always have same truth value)
- Converse and inverse are logically equivalent to each other, but **not** to the original

Example 5: Proving equivalence

Show that $p \to q$ is equivalent to $\sim q \to \sim p$ (contrapositive)

p	q	p → q	~q	~p	~q → ~p
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Since columns for $p \to q$ and $\sim q \to \sim p$ are identical, they are equivalent.

6. Logical Equivalence

Two statements are **logically equivalent** if they have identical truth tables.

Symbol: ≡

Common Equivalences:

De Morgan's Laws:

1. $\sim(p \land q) \equiv \sim p \lor \sim q$
2. $\sim(p \lor q) \equiv \sim p \land \sim q$

Example 6: Verify De Morgan's First Law

p	q	p ∧ q	~(p ∧ q)	~p	~q	~p ∨ ~q
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Columns for $\sim(p \land q)$ and $\sim p \lor \sim q$ are identical. ✓

Other Important Equivalences:

- $p \to q \equiv \sim p \lor q$
- $p \leftrightarrow q \equiv (p \to q) \land (q \to p)$

- $p \land T \equiv p$ (Identity)
- $p \lor F \equiv p$ (Identity)
- $p \land p \equiv p$ (Idempotent)
- $p \lor p \equiv p$ (Idempotent)

7. Tautology, Contradiction, and Contingency

Tautology: A statement that is always true, regardless of truth values of components.

Example: $p \lor \sim p$

p	~p	p ∨ ~p
T	F	T
F	T	T

Contradiction: A statement that is always false.

Example: $p \land \sim p$

p	~p	p ∧ ~p
T	F	F
F	T	F

Contingency: A statement that can be either true or false depending on circumstances.

Example: $p \land q$ (neither tautology nor contradiction)

8. Arguments and Validity

An **argument** consists of:

- **Premises:** Statements assumed to be true
- **Conclusion:** Statement that follows from premises

An argument is **valid** if the conclusion must be true whenever all premises are true.

Example 7: Test validity

Argument:

- Premise 1: $p \to q$
- Premise 2: p
- Conclusion: q

Truth Table:

p	q	p → q	Premises True?	Conclusion
T	T	T	Yes	T
T	F	F	No	-
F	T	T	No	-
F	F	T	No	-

When both premises are true (row 1), conclusion is also true. **This argument is VALID** (Modus Ponens)

Example 8: Invalid argument

Argument:

- Premise 1: $p \rightarrow q$
- Premise 2: q
- Conclusion: p

p	q	p → q	Premises True?	Conclusion
T	T	T	Yes	T
T	F	F	No	-
F	T	T	Yes	F
F	F	T	No	-

In row 3, both premises are true but conclusion is false. **This argument is INVALID** (Fallacy of affirming the consequent)

9. Indirect Proof (Proof by Contradiction)

Indirect proof assumes the opposite of what you want to prove, then shows this leads to a contradiction.

Steps:

1. Assume the negation of the statement to be proved
2. Show this assumption leads to a contradiction
3. Conclude the original statement must be true

Example 9:

Prove: $\sqrt{2}$ is irrational

Proof: Step 1: Assume the opposite: $\sqrt{2}$ is rational Then $\sqrt{2} = a/b$ where a and b are integers with no common factors (lowest terms)

Step 2: Show this leads to contradiction

- $\sqrt{2} = a/b$
- $2 = a^2/b^2$
- $2b^2 = a^2$
- Therefore a^2 is even, so a is even
- Let $a = 2k$ (where k is an integer)
- Then $2b^2 = (2k)^2 = 4k^2$
- $b^2 = 2k^2$
- Therefore b^2 is even, so b is even
- But if both a and b are even, they have a common factor (2)
- This contradicts our assumption that a/b is in lowest terms

Step 3: Conclusion Since assuming $\sqrt{2}$ is rational leads to a contradiction, $\sqrt{2}$ must be irrational. ✓

Example 10: Simple indirect proof

Prove: If n^2 is even, then n is even.

Proof: Step 1: Assume the opposite: n^2 is even but n is odd If n is odd, then $n = 2k + 1$ for some integer k

Step 2: Find contradiction $n^2 = (2k + 1)^2$ $n^2 = 4k^2 + 4k + 1$ $n^2 = 2(2k^2 + 2k) + 1$

This shows n^2 is odd (form $2m + 1$) But we assumed n^2 is even - contradiction!

Step 3: Conclusion Therefore, if n^2 is even, then n must be even. ✓

10. Application: Using Logic in Problem Solving

Example 11:

Three students - Ade, Bola, and Chidi - are suspected of breaking a window. They make the following statements:

- Ade says: "Bola did it"
- Bola says: "Chidi did it"
- Chidi says: "Bola is lying"

If only one person is telling the truth, who broke the window?

Solution:

Let's test each possibility:

Case 1: Ade broke the window

- Ade's statement (Bola did it) - FALSE
- Bola's statement (Chidi did it) - FALSE
- Chidi's statement (Bola is lying) - TRUE ✓

Only Chidi tells the truth. This works!

Case 2: Bola broke the window

- Ade's statement - TRUE
- Bola's statement - FALSE
- Chidi's statement - TRUE

Two people tell truth - doesn't match condition.

Case 3: Chidi broke the window

- Ade's statement - FALSE
- Bola's statement - TRUE
- Chidi's statement - FALSE

Only Bola tells truth, but Bola is the accused - contradiction.

Answer: Ade broke the window

EVALUATION

1. Identify which of the following are statements: a) "Close the door" b) "7 + 5 = 12" c) "Is Lagos a city?" d) "All prime numbers are odd"
2. Write the negation of each statement: a) p: "It is raining" b) q: "5 is greater than 3" c) r: "Some students passed the exam"
3. Construct a truth table for: $p \land \sim q$
4. Construct a truth table for: $(p \lor q) \to r$
5. For the conditional "If you study hard, then you will pass the exam": a) Write the converse b) Write the inverse c) Write the contrapositive
6. Use a truth table to prove that $p \to q$ is equivalent to $\sim p \lor q$
7. Determine whether the following is a tautology, contradiction, or contingency: $(p \to q) \lor (q \to p)$
8. Test the validity of this argument:

- If it rains, the ground is wet
- The ground is wet
- Therefore, it rained

9. Use De Morgan's Law to simplify: $\sim(p \lor \sim q)$

10. Construct a truth table for: $(p \land q) \leftrightarrow (p \lor q)$

ASSIGNMENT

1. **Truth Tables:** a) Construct truth tables for: i) $\sim p \to q$ ii) $(p \to q) \land (\sim q \to \sim p)$ iii) $(p \lor q) \land (\sim p \land \sim q)$

   b) Determine which of the above are tautologies, contradictions, or contingencies.

2. **Logical Equivalence:** Prove using truth tables that: a) $\sim(p \lor q) \equiv \sim p \land \sim q$ (De Morgan's Law) b) $p \to q \equiv \sim q \to \sim p$ c) $(p \to q) \equiv (\sim p \lor q)$

3. **Argument Validity:** Test the validity of these arguments:

   a) Premise 1: $p \lor q$ Premise 2: $\sim p$ Conclusion: q

   b) Premise 1: $p \to q$ Premise 2: $q \to r$ Conclusion: $p \to r$

   c) Premise 1: $p \to q$ Premise 2: $\sim q$ Conclusion: $\sim p$

4. **Word Problem:** Four friends - Amina, Bayo, Cynthia, and David - are discussing who will go to the market. They state:

   - Amina: "If Bayo goes, then I will go"
   - Bayo: "Either Cynthia or David will go"
   - Cynthia: "If I go, then Amina will not go"
   - David: "Bayo and I will both go"

   If all statements are true and exactly two people go to the market, who are they?

5. **Indirect Proof:** Use indirect proof to show that: If n^2 is divisible by 3, then n is divisible by 3.

WEEK 3: APPLICATION OF SURDS TO TRIGONOMETRY

CONTENT

1. Review of Surds

A **surd** is an irrational root (cannot be simplified to a rational number).

Examples of Surds:

- $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$
- $\sqrt[3]{2}$, $\sqrt[3]{3}$
- $2\sqrt{3}$, $5\sqrt{2}$

Examples of Non-Surds:

- $\sqrt{4} = 2$ (rational)
- $\sqrt{9} = 3$ (rational)
- $\sqrt{16} = 4$ (rational)

Basic Surd Rules:

1. $\sqrt{a} \quad \sqrt{b} = \sqrt{(ab)}$ Example: $\sqrt{2} \quad \sqrt{3} = \sqrt{6}$
2. $\sqrt{a} \div \sqrt{b} = \sqrt{(a/b)}$ Example: $\sqrt{8} \div \sqrt{2} = \sqrt{4} = 2$
3. $\sqrt{(a^2b)} = a\sqrt{b}$ Example: $\sqrt{(4 \quad 3)} = \sqrt{12} = 2\sqrt{3}$
4. $(\sqrt{a})^2 = a$ Example: $(\sqrt{5})^2 = 5$

Simplifying Surds:

Example 1: Simplify $\sqrt{18}$ $\sqrt{18} = \sqrt{(9 \times 2)} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$

Example 2: Simplify $\sqrt{50}$ $\sqrt{50} = \sqrt{(25 \times 2)} = 5\sqrt{2}$

Example 3: Simplify $\sqrt{75}$ $\sqrt{75} = \sqrt{(25 \times 3)} = 5\sqrt{3}$

Example 4: Simplify $2\sqrt{12} + 3\sqrt{3}$ $2\sqrt{12} + 3\sqrt{3} = 2\sqrt{(4 \times 3)} + 3\sqrt{3} = 2 \times 2\sqrt{3} + 3\sqrt{3} = 4\sqrt{3} + 3\sqrt{3} = 7\sqrt{3}$

Rationalizing the Denominator:

Example 5: Rationalize $1/\sqrt{2}$ $1/\sqrt{2} = (1/\sqrt{2}) \times (\sqrt{2}/\sqrt{2}) = \sqrt{2}/2$

Example 6: Rationalize $3/\sqrt{5}$ $3/\sqrt{5} = (3/\sqrt{5}) \times (\sqrt{5}/\sqrt{5}) = 3\sqrt{5}/5$

Example 7: Rationalize $1/(2 + \sqrt{3})$ Multiply by conjugate: $1/(2 + \sqrt{3}) \times (2 - \sqrt{3})/(2 - \sqrt{3}) = (2 - \sqrt{3})/(4 - 3) = 2 - \sqrt{3}$

2. Exact Trigonometric Ratios Using Surds

For specific angles (30°, 45°, 60°), we can find **exact** values using surds rather than decimals.

A. Trigonometric Ratios for 45°

Consider a right-angled isosceles triangle with equal sides of length 1:

Using Pythagoras: $\text{hypotenuse}^2 = 1^2 + 1^2 = 2$ Therefore: hypotenuse = $\sqrt{2}$

```
    |\
    | \
  1 |  \ √2
    |   \
    |    \
    |_____\
      1
    45° 45°
```

Exact Values for 45°:

- sin 45° = opposite/hypotenuse = $1/\sqrt{2} = \sqrt{2}/2$
- cos 45° = adjacent/hypotenuse = $1/\sqrt{2} = \sqrt{2}/2$
- tan 45° = opposite/adjacent = 1/1 = 1

B. Trigonometric Ratios for 30° and 60°

Consider an equilateral triangle with sides of length 2, divided in half:

```
      /\
     /  \
   2/    \2
   /  h   \
  /________\
      1  1
  60° 30° 30° 60°
```

Using Pythagoras: $h^2 + 1^2 = 2^2$ $h^2 = 4 - 1 = 3$ $h = \sqrt{3}$

Exact Values for 30°:

- sin 30° = opposite/hypotenuse = 1/2
- cos 30° = adjacent/hypotenuse = $\sqrt{3}/2$
- tan 30° = opposite/adjacent = $1/\sqrt{3} = \sqrt{3}/3$

Exact Values for 60°:

- sin 60° = opposite/hypotenuse = $\sqrt{3}/2$
- cos 60° = adjacent/hypotenuse = 1/2
- tan 60° = opposite/adjacent = $\sqrt{3}/1 = \sqrt{3}$

Summary Table:

Angle	sin	cos	tan
30°	1/2	√3/2	√3/3
45°	√2/2	√2/2	1
60°	√3/2	1/2	√3

Memory Aid (CAST Rule): Remember: **"All Students Take Calculus"**

- **A**ll (0°-90°): All ratios positive
- **S**ine (90°-180°): Only sine positive
- **T**angent (180°-270°): Only tangent positive
- **C**osine (270°-360°): Only cosine positive

3. Worked Examples on Trigonometric Surds

Example 8: Evaluate $\sin^2 45° + \cos^2 45°$

Solution: $\sin^2 45° = (\sqrt{2}/2)^2 = 2/4 = 1/2$ $\cos^2 45° = (\sqrt{2}/2)^2 = 2/4 = 1/2$

$\sin^2 45° + \cos^2 45° = 1/2 + 1/2 = 1$ ✓

(This verifies the identity $\sin^2\theta + \cos^2\theta = 1$)

Example 9: Simplify: $\tan 60° - \tan 30°$

Solution: $\tan 60° = \sqrt{3}$ $\tan 30° = \sqrt{3}/3$

$\tan 60° - \tan 30° = \sqrt{3} - \sqrt{3}/3 = (3\sqrt{3})/3 - \sqrt{3}/3 = (3\sqrt{3} - \sqrt{3})/3 = 2\sqrt{3}/3$

Example 10: Evaluate: $\sin 30° \cos 60° + \cos 30° \sin 60°$

Solution: $\sin 30° = 1/2$, $\cos 60° = 1/2$ $\cos 30° = \sqrt{3}/2$, $\sin 60° = \sqrt{3}/2$

$= (1/2)(1/2) + (\sqrt{3}/2)(\sqrt{3}/2) = 1/4 + 3/4 = 4/4 = 1$

(This is $\sin(30° + 60°) = \sin 90° = 1$) ✓

Example 11: If $\sin \theta = \sqrt{3}/2$ and θ is acute, find: a) $\cos \theta$ b) $\tan \theta$

Solution: a) Using $\sin^2\theta + \cos^2\theta = 1$: $(\sqrt{3}/2)^2 + \cos^2\theta = 1$ $3/4 + \cos^2\theta = 1$ $\cos^2\theta = 1/4$ $\cos \theta = 1/2$ (positive since acute)

b) $\tan \theta = \sin \theta / \cos \theta = (\sqrt{3}/2) \div (1/2) = \sqrt{3}/2 \times 2/1 = \sqrt{3}$

(This confirms $\theta = 60°$)

Example 12: Simplify: $(\sin 60° - \cos 60°)^2$

Solution: $\sin 60° = \sqrt{3}/2$, $\cos 60° = 1/2$

$(\sqrt{3}/2 - 1/2)^2 = [(\sqrt{3} - 1)/2]^2 = (\sqrt{3} - 1)^2/4 = (3 - 2\sqrt{3} + 1)/4 = (4 - 2\sqrt{3})/4 = (2 - \sqrt{3})/2$

Example 13: Solve for x: $2\cos x = \sqrt{3}$, where $0° \leq x \leq 90°$

Solution: $2\cos x = \sqrt{3}$ $\cos x = \sqrt{3}/2$

From our table, $\cos 30° = \sqrt{3}/2$

Answer: $x = 30°$

Example 14: Evaluate: $\sin^2 30° + \sin^2 45° + \sin^2 60°$

Solution: $\sin 30° = 1/2$, $\sin 45° = \sqrt{2}/2$, $\sin 60° = \sqrt{3}/2$

$\sin^2 30° = (1/2)^2 = 1/4$ $\sin^2 45° = (\sqrt{2}/2)^2 = 2/4 = 1/2$ $\sin^2 60° = (\sqrt{3}/2)^2 = 3/4$

Sum $= 1/4 + 1/2 + 3/4 = 1/4 + 2/4 + 3/4 = 6/4 = 3/2$

Example 15: Find the exact value of $\tan 45° + 2\sin 30° - \cos 60°$

Solution: $\tan 45° = 1$ $\sin 30° = 1/2$ $\cos 60° = 1/2$

$= 1 + 2(1/2) - 1/2 = 1 + 1 - 1/2 = 2 - 1/2 = 3/2$

4. Graphs of Sine and Cosine Functions (0° to 360°)

A. Graph of y = sin x

Key Points:

- Period: 360° (pattern repeats every 360°)
- Amplitude: 1 (max height from center)
- Range: $-1 \leq y \leq 1$
- Passes through origin: $\sin 0° = 0$

Special Values:

```
x:    0°    30°   45°   60°   90°   180°  270°  360°
y:    0     0.5   0.71  0.87  1      0     -1     0
```

Shape:

- Starts at 0° with value 0
- Increases to maximum 1 at 90°
- Decreases back to 0 at 180°
- Continues to minimum -1 at 270°
- Returns to 0 at 360°

Sketch:

```
   1 |     ___
     |    /   \
   0 |___/     \___
     |             \  /
  -1 |              \/
     |________________
     0°  90° 180° 270° 360°
```

B. Graph of y = cos x

Key Points:

- Period: 360°
- Amplitude: 1
- Range: $-1 \leq y \leq 1$
- Starts at maximum: cos 0° = 1

Special Values:

```
x:    0°    30°   60°   90°   180°  270°  360°
y:    1    0.87  0.5    0      -1    0      1
```

Shape:

- Starts at 0° with value 1 (maximum)
- Decreases to 0 at 90°
- Continues to minimum -1 at 180°
- Increases to 0 at 270°
- Returns to maximum 1 at 360°

Sketch:

```
   1 |___       ___
     |   \     /
   0 |    \___/
     |     \ /
  -1 |      \/
```

```
|________________
0°  90° 180° 270° 360°
```

Relationship: cos x = sin(x + 90°) The cosine graph is the sine graph shifted 90° to the left.

C. Properties of Sine and Cosine Graphs

Symmetry:

- sin x is odd function: sin(-x) = -sin(x)
- cos x is even function: cos(-x) = cos(x)

Periodicity:

- sin(x + 360°) = sin x
- cos(x + 360°) = cos x

Intersections:

- Sine and cosine graphs intersect at x = 45°, 225°

5. Using Graphs to Solve Trigonometric Equations

Example 16: Use a graph to solve sin x = 0.5 for 0° ≤ x ≤ 360°

Solution: Draw y = sin x and y = 0.5 on the same axes.

From the graph and knowledge: sin 30° = 0.5 (first quadrant) sin 150° = 0.5 (second quadrant - supplementary angle)

Answer: x = 30° or x = 150°

Verification:

- sin 30° = 1/2 = 0.5 ✓
- sin 150° = sin(180° - 30°) = sin 30° = 0.5 ✓

Example 17: Solve cos x = 0.5 for 0° ≤ x ≤ 360°

Solution: From the table: cos 60° = 0.5

Using the cosine graph:

- cos x = 0.5 in first quadrant: x = 60°
- cos x = 0.5 in fourth quadrant: x = 360° - 60° = 300°

Answer: x = 60° or x = 300°

Example 18: Solve sin x = cos x for $0° \leq x \leq 360°$

Solution: This occurs where sine and cosine graphs intersect.

sin x = cos x sin x / cos x = 1 tan x = 1

tan 45° = 1, so x = 45° (first quadrant) tan 225° = 1, so x = 225° (third quadrant)

Answer: x = 45° or x = 225°

Example 19: Solve graphically: sin x = $\sqrt{3}/2$ for $0° \leq x \leq 360°$

Solution: $\sqrt{3}/2 \approx 0.866$

From table: sin 60° = $\sqrt{3}/2$

Solutions:

- First quadrant: x = 60°
- Second quadrant: x = 180° - 60° = 120°

Answer: x = 60° or x = 120°

6. Solving Linear Equations Using Graphs

Example 20: Solve graphically: 2x + 3 = 0

Solution: Step 1: Rearrange to y-form: y = 2x + 3

Step 2: Find two points: When x = 0: y = 3 → (0, 3) When x = -2: y = -1 → (-2, -1)

Step 3: Plot line and find where y = 0

The line crosses x-axis at x = -1.5

Answer: x = -1.5

Verification: 2(-1.5) + 3 = -3 + 3 = 0 ✓

Example 21: Solve simultaneously using graphs: y = 2x + 1 y = -x + 4

Solution: Line 1: y = 2x + 1 When x = 0: y = 1 → (0, 1) When x = 1: y = 3 → (1, 3)

Line 2: y = -x + 4 When x = 0: y = 4 → (0, 4) When x = 4: y = 0 → (4, 0)

Plot both lines. They intersect at (1, 3).

Answer: x = 1, y = 3

Verification: Line 1: y = 2(1) + 1 = 3 ✓ Line 2: y = -(1) + 4 = 3 ✓

Example 22: Solve graphically: sin x = 0.5x for $0° \le x \le 180°$

Solution: Draw y = sin x and y = 0.5x on same axes.

The graphs intersect at two points:

- x = 0° (both equal 0)
- x ≈ 30° (approximate from graph)

Note: For exact solution, numerical methods would be needed.

EVALUATION

1. Simplify: (a) $\sqrt{32}$ (b) $\sqrt{45}$ (c) $3\sqrt{8} + 2\sqrt{2}$
2. Rationalize: (a) $1/\sqrt{3}$ (b) $2/\sqrt{5}$ (c) $1/(1 + \sqrt{2})$
3. Evaluate exactly: $\sin^2 30° + \cos^2 30°$
4. Find the exact value of: tan 60°   cos 30°
5. Simplify: $(\sin 45° + \cos 45°)^2$
6. Solve for x ($0° \leq x \leq 90°$): 2sin x = 1
7. Evaluate: sin 30° cos 60° - cos 30° sin 60°
8. Sketch the graph of y = sin x for $0° \leq x \leq 360°$
9. Use a graph to solve: cos x = 0 for $0° \leq x \leq 360°$
10. If tan θ = $\sqrt{3}$ and θ is acute, find sin θ and cos θ.

ASSIGNMENT

1. **Simplification:** a) Simplify completely: $\sqrt{48} + \sqrt{75} - \sqrt{12}$ b) Rationalize and simplify: $3/(\sqrt{5} - \sqrt{2})$ c) Simplify: $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$
2. **Exact Trigonometric Values:** a) Evaluate: $\sin^2 45° + \cos^2 60° - \tan^2 30°$ b) Find the exact value of: (sin 30° + cos 60°)/(tan 45°) c) Simplify: 2sin 60° cos 30° - sin 90°
3. **Equations:** a) Solve for θ ($0° \leq θ \leq 360°$): sin θ = $\sqrt{2}/2$ b) Solve: 2cos θ -1 = 0 for $0° \leq θ \leq 360°$ c) Find all values of x ($0° \leq x \leq 360°$) for which: tan x = $1/\sqrt{3}$

4. **Graphical Solutions:** a) Draw the graph of $y = \cos x$ for $0° \leqslant x \leqslant 360°$, taking 1 cm = 30° on x-axis and 2 cm = 1 unit on y-axis. b) Use your graph to solve: $\cos x = 0.5$ c) Use your graph to find approximate solutions to: $\cos x = 0.7$

5. **Challenge Problem:** Prove that: $(\sin 60° - \sin 30°)/(\cos 30° - \cos 60°) = \sqrt{3}$

WEEK 4: MATRICES AND DETERMINANTS

CONTENT

1. Definition of a Matrix

A **matrix** is a rectangular array of numbers arranged in rows and columns, enclosed in brackets.

General Form:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Where:

- $\mathbf{a_{ij}}$ represents the element in the **i**th row and **j**th column
- **Rows** run horizontally
- **Columns** run vertically

Example:

$$M = \begin{bmatrix} 2 & 3 & -1 \\ 5 & 0 & 4 \\ 1 & -2 & 6 \end{bmatrix}$$

- Element $a_{12} = 3$ (row 1, column 2)
- Element $a_{23} = 4$ (row 2, column 3)
- Element $a_{31} = 1$ (row 3, column 1)

2. Order of a Matrix

The **order** (or dimension) of a matrix is given as **m × n** (read as "m by n"), where:

- **m** = number of rows
- **n** = number of columns

Example 1:

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \\ 4 & -2 \end{bmatrix}$$

Order: **3 × 2** (3 rows, 2 columns)

Example 2:

$$B = \begin{bmatrix} 1 & 0 & 3 & 5 \\ 2 & -1 & 4 & 6 \end{bmatrix}$$

Order: **2 × 4** (2 rows, 4 columns)

Example 3:

```
C = [   4 ]
    |   7 |
    [   2 ]
```

Order: **3** × **1** (column matrix)

Example 4:

```
D = [ 3   5   -2   1 ]
```

Order: **1** × **4** (row matrix)

3. Types of Matrices

A. Row Matrix A matrix with only **one row**.

Example:

```
R = [ 5   3   -1   8 ]    Order: 1 × 4
```

B. Column Matrix A matrix with only **one column**.

Example:

```
C = [   2 ]
    |   5 |      Order: 3 × 1
    [   7 ]
```

C. Square Matrix A matrix where **number of rows = number of columns** (m = n).

Example:

```
S = [   1   2   3 ]
    |   4   5   6 |      Order: 3 × 3
    [   7   8   9 ]
```

D. Diagonal Matrix A square matrix where all elements are zero except possibly those on the main diagonal.

Example:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

E. Identity Matrix (Unit Matrix) A diagonal matrix where all diagonal elements are 1. Denoted by **I**.

Examples:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Property: For any matrix A, **A × I = I × A = A**

F. Zero Matrix (Null Matrix) A matrix where all elements are zero. Denoted by **O** or **0**.

Example:

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

G. Scalar Matrix A diagonal matrix where all diagonal elements are equal.

Example:

$$S = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

H. Upper Triangular Matrix A square matrix where all elements below the main diagonal are zero.

Example:

$$U = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

I. Lower Triangular Matrix A square matrix where all elements above the main diagonal are zero.

Example:

$$L = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 2 & 0 \\ 1 & 4 & 6 \end{bmatrix}$$

4. Matrix Operations

A. Equality of Matrices

Two matrices are equal if:

1. They have the same order
2. Corresponding elements are equal

Example 5: If

$$\begin{bmatrix} x & 2 \\ 4 & y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$$

Then x = 3 and y = 5

B. Addition of Matrices

Matrices can only be added if they have the **same order**. Add corresponding elements.

Example 6:

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2+1 & 3+4 \\ 5+2 & 1+3 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 4 \end{bmatrix}$$

Example 7:

$$\begin{bmatrix} 3 & -1 & 2 \\ 4 & 0 & 5 \\ 1 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 4 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 3 \\ 5 & -2 & 9 \\ 4 & 3 & -1 \end{bmatrix}$$

Properties of Addition:

1. **Commutative:** A + B = B + A
2. **Associative:** (A + B) + C = A + (B + C)
3. **Identity:** A + O = A (O is zero matrix)

C. Subtraction of Matrices

Subtract corresponding elements (matrices must have same order).

Example 8:

$$\begin{bmatrix} 5 & 3 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5-2 & 3-1 \\ 4-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$$

Example 9:

$$\begin{bmatrix} 7 & 2 \\ 5 & -3 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 2 & 5 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & -8 \\ -3 & 4 \end{bmatrix}$$

D. Scalar Multiplication

Multiply every element of the matrix by the scalar (constant).

Example 10:

$$3 \times \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = 3 \times \begin{bmatrix} 6 & 9 \\ 3 & 12 \end{bmatrix}$$

Example 11:

$$-2 \times \begin{bmatrix} 4 & -2 \\ 3 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -8 & 4 \\ -6 & 0 \\ -2 & -10 \end{bmatrix}$$

E. Matrix Multiplication

Two matrices A (order m × n) and B (order p × q) can be multiplied if **n = p** (number of columns in A = number of rows in B).

The resulting matrix has order **m × q**.

Rule: Element in row i, column j of AB = (row i of A) · (column j of B)

Example 12: Simple 2×2 multiplication

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

Solution:

```
Element (1,1) = (2×1) + (3×2) = 2 + 6 = 8
Element (1,2) = (2×4) + (3×3) = 8 + 9 = 17
Element (2,1) = (5×1) + (1×2) = 5 + 2 = 7
Element (2,2) = (5×4) + (1×3) = 20 + 3 = 23
```

Answer:

$$\begin{bmatrix} 8 & 17 \\ 7 & 23 \end{bmatrix}$$

Example 13:

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Solution: Order check: (2×2) × (2×1) = (2×1) ✓

$$\begin{bmatrix} (1\times3)+(2\times1) \\ (4\times3)+(5\times1) \end{bmatrix} = \begin{bmatrix} 5 \\ 17 \end{bmatrix}$$

Example 14:

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 4 \end{bmatrix}$$

Solution: Order: (2×3) × (3×2) = (2×2) ✓

Element (1,1) = (2×1) + (1×3) + (3×2) = 2 + 3 + 6 = 11
Element (1,2) = (2×2) + (1×1) + (3×4) = 4 + 1 + 12 = 17
Element (2,1) = (4×1) + (0×3) + (2×2) = 4 + 0 + 4 = 8
Element (2,2) = (4×2) + (0×1) + (2×4) = 8 + 0 + 8 = 16

Answer:

$$\begin{bmatrix} 11 & 17 \\ 8 & 16 \end{bmatrix}$$

Important: Matrix multiplication is **NOT commutative**: AB ≠ BA (in general)

Example 15: Verify that AB ≠ BA

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$$

AB:

$$\begin{bmatrix} (1\times3)+(2\times2) & (1\times1)+(2\times5) \\ (3\times3)+(4\times2) & (3\times1)+(4\times5) \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 17 & 23 \end{bmatrix}$$

BA:

$$\begin{bmatrix} (3\times1)+(1\times3) & (3\times2)+(1\times4) \\ (2\times1)+(5\times3) & (2\times2)+(5\times4) \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ 17 & 24 \end{bmatrix}$$

Clearly AB ≠ BA

Properties of Matrix Multiplication:

1. **Associative:** (AB)C = A(BC)
2. **Distributive:** A(B + C) = AB + AC
3. **Identity:** AI = IA = A
4. **NOT Commutative:** AB ≠ BA (generally)

5. Transpose of a Matrix

The **transpose** of a matrix A, denoted $\mathbf{A^T}$ or **A'**, is obtained by interchanging rows and columns.

Rule: If $A = [a_{ij}]$, then $A^T = [a_{ji}]$

Example 16:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Example 17:

$$B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

Example 18:

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Properties of Transpose:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(kA)^T = kA^T$ (k is scalar)

4. $(AB)^T = B^T A^T$ (note the reversal)

Symmetric Matrix: A matrix where $A = A^T$

Example:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 4 & 6 \\ 5 & 6 & 7 \end{bmatrix} \quad \text{(Symmetric)}$$

6. Determinants

The **determinant** is a scalar value calculated from a square matrix. Denoted **det(A)** or **|A|**.

A. Determinant of 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = |A| = ad - bc$$

Example 19:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$|A| = (3\times4) - (2\times1) = 12 - 2 = 10$$

Example 20:

$$B = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$$

$$|B| = (5\times1) - (-2\times3) = 5 + 6 = 11$$

Example 21:

$$C = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$$

$$|C| = (2\times3) - (6\times1) = 6 - 6 = 0$$

(This matrix is **singular** - has no inverse)

B. Determinant of 3×3 Matrix

For a 3×3 matrix, use the **rule of Sarrus** or **expansion by cofactors**.

Method 1: Rule of Sarrus

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Copy first two columns to the right:

$$\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

Positive diagonals: aei + bfg + cdh **Negative diagonals:** ceg + afh + bdi

|A| = (aei + bfg + cdh) - (ceg + afh + bdi)

Example 22:

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 5 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution: Positive: $(2\times4\times1) + (1\times5\times1) + (3\times0\times2) = 8 + 5 + 0 = 13$ Negative: $(3\times4\times1) + (2\times5\times2) + (1\times0\times1) = 12 + 20 + 0 = 32$

$|A| = 13 - 32 = -19$

Method 2: Expansion by First Row (Cofactor Method)

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$|A| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Example 23:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Solution:

$$|A| = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1[(5\times9)-(6\times8)] - 2[(4\times9)-(6\times7)] + 3[(4\times8)-(5\times7)]$$
$$= 1[45-48] - 2[36-42] + 3[32-35]$$

$$= 1(-3) - 2(-6) + 3(-3)$$
$$= -3 + 12 - 9$$
$$= 0$$

(Rows are dependent - matrix is singular)

Example 24:

$$B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 3 & 0 & 1 \end{bmatrix}$$

Solution:

$$|B| = 2\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$$

$$= 2[(2\times1)-(4\times0)] + 1[(1\times1)-(4\times3)] + 3[(1\times0)-(2\times3)]$$
$$= 2(2-0) + 1(1-12) + 3(0-6)$$
$$= 2(2) + 1(-11) + 3(-6)$$
$$= 4 - 11 - 18$$
$$= -25$$

Properties of Determinants:

1. If any row or column is all zeros, $|A| = 0$
2. If two rows (or columns) are identical, $|A| = 0$
3. $|A^T| = |A|$
4. $|AB| = |A| \times |B|$
5. $|kA| = k^n|A|$ (n = order of matrix)
6. Interchanging two rows changes sign: $|A| \to -|A|$

7. Inverse of a Matrix

The **inverse** of a square matrix A (if it exists) is denoted $\mathbf{A^{-1}}$, such that: $\mathbf{A \times A^{-1} = A^{-1} \times A = I}$

Conditions for Inverse to Exist:

- Matrix must be **square**
- $\mathbf{|A| \neq 0}$ (non-singular)

A. Inverse of 2×2 Matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Steps:

1. Calculate |A| = ad - bc
2. If |A| = 0, inverse doesn't exist
3. Swap diagonal elements (a↔d)
4. Change signs of off-diagonal elements (-b, -c)
5. Divide by |A|

Example 25:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Solution: Step 1: |A| = (2×4) - (3×1) = 8 - 3 = 5

Step 2: Swap and change signs:

$$\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Step 3: Divide by |A|:

$$A^{-1} = \frac{1}{5} \times \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix}$$

Verification:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \times \begin{bmatrix} 4/5 & -3/5 \\ -1/5 & 2/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Example 26:

$$B = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix}$$

Solution: |B| = (5×1) - (2×3) = 5 - 6 = -1

$$B^{-1} = \frac{1}{-1} \begin{bmatrix} 1 & -2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & -5 \end{bmatrix}$$

Example 27: Matrix with no inverse

$$C = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

$|C| = (2\times2) - (4\times1) = 4 - 4 = 0$

No inverse exists (singular matrix)

8. Solving Simultaneous Equations Using Matrices

Method: If we have equations in matrix form **AX = B**, then $\mathbf{X = A^{-1}B}$

Example 28: Solve:

$$2x + y = 5$$
$$x + 3y = 8$$

Solution:

Step 1: Matrix form

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Step 2: Find A^{-1} $|A| = (2\times3) - (1\times1) = 6 - 1 = 5$

$$A^{-1} = (1/5) \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$$

Step 3: $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

$$x = (3/5)(5) + (-1/5)(8) = 15/5 - 8/5 = 7/5$$
$$y = (-1/5)(5) + (2/5)(8) = -5/5 + 16/5 = 11/5$$

Verification: $2(7/5) + (11/5) = 14/5 + 11/5 = 25/5 = 5$ ✓ $(7/5) + 3(11/5) = 7/5 + 33/5 = 40/5 = 8$ ✓

Answer: x = 7/5 = 1.4, y = 11/5 = 2.2

Example 29: Solve using matrices:

$$3x + 2y = 7$$
$$x + y = 3$$

Solution:

Matrix form:

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$|A| = (3\times1) - (2\times1) = 3 - 2 = 1$

$$A^{-1} = (1/1)\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} (1)(7) + (-2)(3) \\ (-1)(7) + (3)(3) \end{bmatrix} = \begin{bmatrix} 7-6 \\ -7+9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Verification: $3(1) + 2(2) = 3 + 4 = 7$ ✓ $1 + 2 = 3$ ✓

Answer: $x = 1, y = 2$

EVALUATION

1. State the order of the following matrices: a) A = [1 2 3] [4 5 6]

   b) B = [2] | 5 | [7]
2. Given A = [2 3] and B = [1 4] , find: [1 5] [2 3] a) A + B b) A - B c) 2A
3. Find AB if A = [1 2] and B = [3 1] [3 4] [2 5]
4. Find the transpose of A = [2 1 3] [4 5 6]
5. Calculate the determinant: |2 3| |1 4|
6. Find the determinant: |1 2 3| |0 4 5| |1 2 1|
7. Find the inverse of A = [3 1] [2 1]
8. Find the inverse of B = [1 2] [3 4]
9. Solve using matrices: 2x + y = 7 x + y = 4
10. Identify the type of matrix: [1 0 0] | 0 1 0 | [0 0 1]

ASSIGNMENT

1. **Matrix Operations:** Given A = [2 -1] , B = [3 2] , and C = [1 0] | 3 4 | | 1 5 | | 2 3 | [0 2] [4 1] [1 2]

   Calculate: a) 2A - 3B b) AB c) $(A + B)^T$ d) BC
2. **Determinants:** a) Find |A| if A = [4 2] [-3 1]

   b) Calculate: |2 1 3| |1 0 2| |4 2 1|

   c) Find the value of k if |k 2| = 0 |3 6|

3. **Matrix Inverses:** a) Find the inverse of M = [5 3] [2 1]

   b) Verify that M M⁻ ¹ = I

   c) Find N⁻ ¹ if N = [4 1] [3 1]

4. **Solving Systems:** Use matrices to solve: a) 3x + 2y = 11 2x + y = 7

   b) x + 3y = 5 2x + y = 4

   c) 4x - y = 10 x + 2y = 5

5. **Application Problem:** A company produces two products A and B. The production requirements are given in the matrix:

```
      [  2   3 ]   (hours of labor)
P =   [  1   2 ]   (units of material)
```

   Where columns represent products A and B.

   a) If the company produces 10 units of A and 15 units of B, write a matrix equation to find total labor hours and materials needed. b) Calculate the totals. c) If labor costs ₦500 per hour and material costs ₦200 per unit, find the total cost using matrix multiplication.

WEEK 6: LINEAR AND QUADRATIC EQUATIONS

CONTENT

1. Review: Linear Equations

A **linear equation** has the form **$y = mx + c$** where:

- **m** = slope (gradient)
- **c** = y-intercept
- Graph is a **straight line**

Example 1: $y = 2x + 3$

- Slope $m = 2$
- y-intercept $c = 3$

2. Review: Quadratic Equations

A **quadratic equation** has the form **$y = ax^2 + bx + c$** where $a \neq 0$:

- **a, b, c** are constants
- **a** determines if parabola opens up ($a > 0$) or down ($a < 0$)
- Graph is a **parabola**

Example 2: $y = x^2 - 4x + 3$

- $a = 1$ (opens upward)
- $b = -4$
- $c = 3$

3. Solving Simultaneous Linear and Quadratic Equations Algebraically

Method: Substitution

Example 3: Solve simultaneously:

$$y = x + 2 \quad \ldots \text{ (1) [linear]}$$
$$y = x^2 - 4 \quad \ldots \text{ (2) [quadratic]}$$

Solution:

Step 1: Substitute equation (1) into equation (2)

$$x + 2 = x^2 - 4$$

Step 2: Rearrange to standard form

$$x^2 - x - 6 = 0$$

Step 3: Factorize

$(x - 3)(x + 2) = 0$
$x = 3$ or $x = -2$

Step 4: Find corresponding y-values using equation (1)

When $x = 3$: $y = 3 + 2 = 5$
When $x = -2$: $y = -2 + 2 = 0$

Solutions: (3, 5) and (-2, 0)

Verification: For (3, 5):

- Linear: $y = 3 + 2 = 5$ ✓
- Quadratic: $y = 9 - 4 = 5$ ✓

For (-2, 0):

- Linear: $y = -2 + 2 = 0$ ✓
- Quadratic: $y = 4 - 4 = 0$ ✓

Example 4: Solve:

$y = 2x - 1 \quad \ldots (1)$
$y = x^2 + x - 5 \ldots (2)$

Solution:

Substitute (1) into (2):

$2x - 1 = x^2 + x - 5$
$0 = x^2 + x - 2x - 5 + 1$
$x^2 - x - 4 = 0$

Using quadratic formula (doesn't factorize nicely):

$x = [1 \pm \sqrt{(1 + 16)}] / 2$
$x = [1 \pm \sqrt{17}] / 2$
$x = (1 + \sqrt{17})/2$ or $x = (1 - \sqrt{17})/2$
$x \approx 2.56$ or $x \approx -1.56$

Find y-values:

When $x \approx 2.56$: $y = 2(2.56) - 1 \approx 4.12$
When $x \approx -1.56$: $y = 2(-1.56) - 1 \approx -4.12$

Solutions: (2.56, 4.12) and (-1.56, -4.12) (approximate)

Example 5: Solve:

$$x + y = 5 \quad \ldots (1)$$
$$x^2 + y^2 = 13 \quad \ldots (2)$$

Solution:

From (1): $y = 5 - x$

Substitute into (2):

$$x^2 + (5 - x)^2 = 13$$
$$x^2 + 25 - 10x + x^2 = 13$$
$$2x^2 - 10x + 25 = 13$$
$$2x^2 - 10x + 12 = 0$$
$$x^2 - 5x + 6 = 0$$
$$(x - 2)(x - 3) = 0$$
$$x = 2 \text{ or } x = 3$$

When $x = 2$: $y = 5 - 2 = 3$ When $x = 3$: $y = 5 - 3 = 2$

Solutions: (2, 3) and (3, 2)

Example 6: Solve:

$$y = x - 1 \quad \ldots (1)$$
$$y = x^2 - 5x + 5 \quad \ldots (2)$$

Solution:

Substitute:

$$x - 1 = x^2 - 5x + 5$$
$$0 = x^2 - 5x - x + 5 + 1$$
$$x^2 - 6x + 6 = 0$$

Using quadratic formula:

$$x = [6 \pm \sqrt{(36 - 24)}] / 2$$
$$x = [6 \pm \sqrt{12}] / 2$$
$$x = [6 \pm 2\sqrt{3}] / 2$$
$$x = 3 \pm \sqrt{3}$$
$$x = 3 + \sqrt{3} \text{ or } x = 3 - \sqrt{3}$$
$$x \approx 4.73 \text{ or } x \approx 1.27$$

Find y-values:

When $x \approx 4.73$: $y \approx 3.73$
When $x \approx 1.27$: $y \approx 0.27$

Solutions: (4.73, 3.73) and (1.27, 0.27) (approximate)

4. Solving Graphically

Steps:

1. Draw both graphs on the same axes
2. Identify points of intersection
3. Read coordinates from graph

Example 7: Solve graphically:

```
y = x + 1
y = x² - 2x
```

Solution:

For $y = x + 1$ (straight line):

```
x:  -2   -1    0    1    2
y:  -1    0    1    2    3
```

For $y = x^2 - 2x$ (parabola):

```
x:  -2   -1    0    1    2    3    4
y:   8    3    0   -1    0    3    8
```

Plot both curves. They intersect at approximately:

- **(-1, 0)**
- **(3, 4)**

Example 8: Interpret number of solutions from graphs

Case 1: Line intersects parabola at two points → **Two solutions**

Case 2: Line is tangent to parabola (touches at one point) → **One solution** (repeated root)

Case 3: Line doesn't touch parabola → **No real solutions**

5. Word Problems

Example 9: The sum of two numbers is 10, and their product is 21. Find the numbers.

Solution:

Let the numbers be x and y.

```
x + y = 10  ... (1)
xy = 21     ... (2)
```

From (1): $y = 10 - x$

Substitute into (2):

```
x(10 - x) = 21
10x - x² = 21
x² - 10x + 21 = 0
(x - 3)(x - 7) = 0
x = 3 or x = 7
```

When $x = 3$: $y = 7$ When $x = 7$: $y = 3$

The numbers are 3 and 7

Example 10: A rectangular field has length 5m more than its width. If the area is 84 m², find the dimensions.

Solution:

Let width = x meters Then length = (x + 5) meters

Area equation:

```
x(x + 5) = 84
x² + 5x = 84
x² + 5x - 84 = 0
(x + 12)(x - 7) = 0
x = -12 or x = 7
```

Since width must be positive: $x = 7$ m

Length = 7 + 5 = 12 m

Dimensions: 7 m × 12 m

Verification: 7 × 12 = 84 ✓

Example 11 (Capital Market Application):

An investor buys shares at ₦x each. The number of shares bought is (100 - x). If the total investment is ₦2,400, find: a) The equation relating x and the number of shares b) The price per share c) The number of shares bought

Solution:

a) Price per share × Number of shares = Total investment

```
x(100 - x) = 2400
```

b) Solve for x:

$$100x - x^2 = 2400$$
$$x^2 - 100x + 2400 = 0$$
$$(x - 40)(x - 60) = 0$$
$$x = 40 \text{ or } x = 60$$

c) If $x = 40$: Number of shares = 100 - 40 = 60 shares If $x = 60$: Number of shares = 100 - 60 = 40 shares

Two possible scenarios:

- **₦40 per share, 60 shares**, or
- **₦60 per share, 40 shares**

Example 12: A trader bought x items for ₦$(x^2 + 2x)$ and sold them for ₦$(3x^2 - 4x)$. If the profit was ₦140, find the number of items.

Solution:

Profit = Selling Price - Cost Price

$$(3x^2 - 4x) - (x^2 + 2x) = 140$$
$$3x^2 - 4x - x^2 - 2x = 140$$
$$2x^2 - 6x = 140$$
$$2x^2 - 6x - 140 = 0$$
$$x^2 - 3x - 70 = 0$$
$$(x - 10)(x + 7) = 0$$
$$x = 10 \text{ or } x = -7$$

Since number of items must be positive: **x = 10 items**

Verification: Cost = $10^2 + 2(10) = 100 + 20 =$ ₦120 Selling Price = $3(10^2) - 4(10) = 300 - 40 =$ ₦260 Profit = 260 - 120 = ₦140 ✓

6. Interpreting Quadratic Graphs

Key Features:

A. Vertex (Turning Point)

- Maximum (if parabola opens down)
- Minimum (if parabola opens up)
- Found using $x = -b/(2a)$

B. Axis of Symmetry

- Vertical line through vertex
- Equation: $x = -b/(2a)$

C. Y-intercept

- Where graph crosses y-axis
- Point: (0, c)

D. X-intercepts (Roots)

- Where graph crosses x-axis
- Found by solving $ax^2 + bx + c = 0$

Example 13: For $y = x^2 - 4x + 3$, find: a) Vertex b) Axis of symmetry c) Y-intercept d) X-intercepts

Solution:

a) Vertex:

```
x = -b/(2a) = -(-4)/(2×1) = 4/2 = 2
y = (2)² - 4(2) + 3 = 4 - 8 + 3 = -1
Vertex: (2, -1)
```

b) Axis of symmetry: $\mathbf{x = 2}$

c) Y-intercept: Put $x = 0$

```
y = 0 - 0 + 3 = 3
Y-intercept: (0, 3)
```

d) X-intercepts: Put $y = 0$

```
x² - 4x + 3 = 0
(x - 1)(x - 3) = 0
x = 1 or x = 3
X-intercepts: (1, 0) and (3, 0)
```

EVALUATION

1. Solve simultaneously: $y = x + 3$ $y = x^2 + x - 2$
2. Solve algebraically: $y = 2x - 1$ $y = x^2 - 3$
3. Solve: $x + y = 6$ $xy = 8$
4. The sum of two numbers is 12 and their product is 35. Find the numbers.
5. A rectangular garden has length 3m more than its width. If the area is 40 m², find the dimensions.
6. For the equation $y = x^2 - 6x + 5$, find: a) The vertex b) The axis of symmetry c) The y-intercept d) The x-intercepts
7. Solve: $y = x - 2$ $y = x^2 - 4x + 2$
8. How many solutions does the system have if the discriminant of the resulting quadratic is negative?

9. An investor buys shares at ₦x each. The number bought is $(80 - 2x)$. If total investment is ₦800, find the price per share.

10. Sketch the graphs of $y = 2x$ and $y = x^2 - 4$ on the same axes. Estimate the solutions graphically.

ASSIGNMENT

1. **Algebraic Solutions:** Solve the following systems: a) $y = x + 1$ $y = x^2 - 3x + 2$

   b) $y = 3x - 2$ $y = 2x^2 - x - 1$

   c) $x - y = 2$ $x^2 + y^2 = 10$

   d) $y = 2x$ $y = x^2 - 8$

2. **Word Problems:** a) The difference between two numbers is 4 and their product is 45. Find the numbers.

   b) A rectangular playground has perimeter 36 m and area 80 m^2. Find its dimensions.

   c) The sum of a number and its reciprocal is 2.9. Find the number.

   d) A ball is thrown upward. Its height h (in meters) after t seconds is given by: $h = 20t - 5t^2$ i) When does the ball hit the ground? ii) What is the maximum height reached?

3. **Capital Market Application:** A stockbroker bought x shares at ₦$(200 - 2x)$ each. a) Write an expression for the total cost b) If the total cost was ₦9,600, form an equation and solve for x c) Find the price per share d) How many shares were bought?

4. **Graphical Solution:** a) On the same axes, draw the graphs of: $y = x + 2$ (for $-3 \leqslant x \leqslant 5$) $y = x^2 - 4x$ (for $-1 \leqslant x \leqslant 5$) Use 2 cm = 1 unit on both axes

   b) From your graph, find the solutions to: $x + 2 = x^2 - 4x$

   c) Verify your graphical solutions algebraically

5. **Analysis:** For the quadratic function $y = -x^2 + 6x - 5$: a) Find the vertex and state whether it's a maximum or minimum b) Find the y-intercept c) Find the x-intercepts d) Sketch the graph e) State the range of values of x for which y is positive

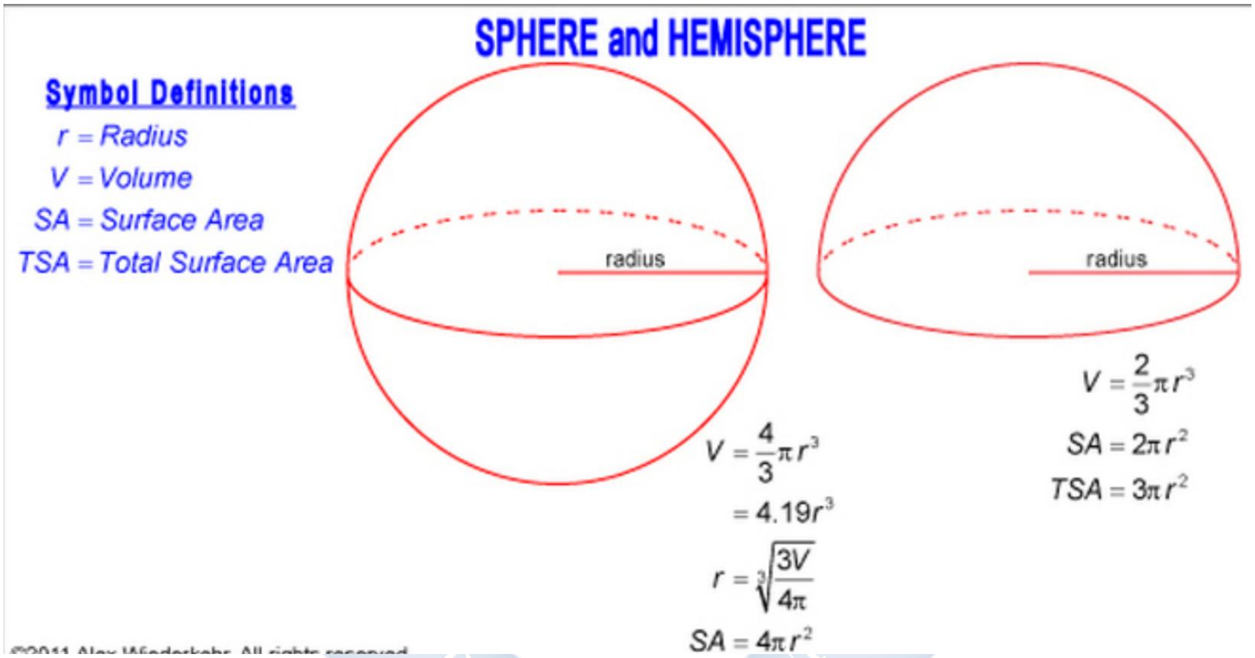
---

WEEK 8: SURFACE AREA AND VOLUME OF SPHERES AND HEMISPHERES

CONTENT

1. Introduction to Spheres

A **sphere** is a perfectly round three-dimensional solid object where every point on its surface is equidistant from its center.



Key Terms:

- **Radius (r):** Distance from center to any point on the surface
- **Diameter (d):** Distance across the sphere through the center (d = 2r)
- **Center:** The fixed point equidistant from all surface points

Examples in Real Life:

- Football, basketball, tennis ball
- Globe, planets, moon
- Oranges, watermelons
- Balloons (when inflated spherically)
- Marbles, beads

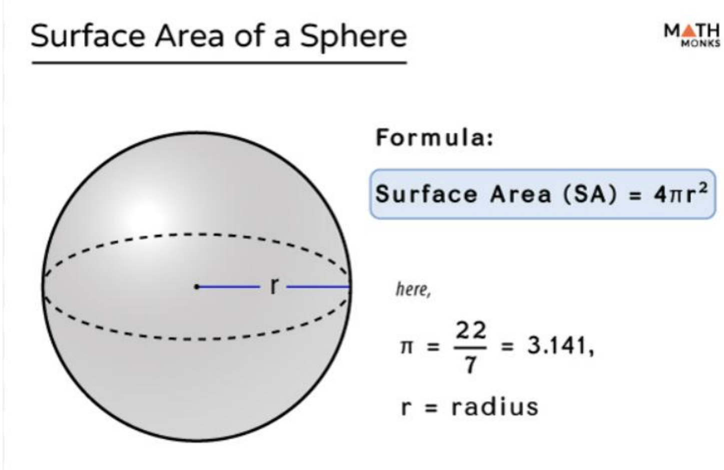
2. Surface Area of a Sphere

The **surface area** is the total area covering the outside of the sphere.

Formula: Surface Area of Sphere = $4\pi r^2$

Where:

- r = radius
- π = 22/7 or 3.142



Derivation Concept: The surface area of a sphere is exactly **4 times** the area of a circle with the same radius.

- Area of circle = πr^2
- Surface area of sphere = $4\pi r^2$

Example 1: Find the surface area of a sphere with radius 7 cm.

Solution:

```
Surface Area = 4πr²
= 4 × (22/7) × 7²
= 4 × (22/7) × 49
= 4 × 22 × 7
= 616 cm²
```

Answer: 616 cm²

Example 2: A spherical balloon has radius 14 cm. Find its surface area.

Solution:

```
Surface Area = 4πr²
= 4 × (22/7) × 14²
= 4 × (22/7) × 196
= 4 × 22 × 28
= 2,464 cm²
```

Answer: 2,464 cm²

Example 3: The surface area of a sphere is 1,386 cm². Calculate its radius. (Use $\pi = 22/7$)

Solution:

$$4\pi r^2 = 1{,}386$$
$$4 \times (22/7) \times r^2 = 1{,}386$$
$$(88/7) \times r^2 = 1{,}386$$
$$r^2 = 1{,}386 \times (7/88)$$
$$r^2 = 1{,}386 \div 88 \times 7$$
$$r^2 = 15.75 \times 7$$
$$r^2 = 110.25$$
$$r = \sqrt{110.25}$$
$$r = 10.5 \text{ cm}$$

Answer: r = 10.5 cm

Example 4: A sphere has diameter 21 cm. Find its surface area.

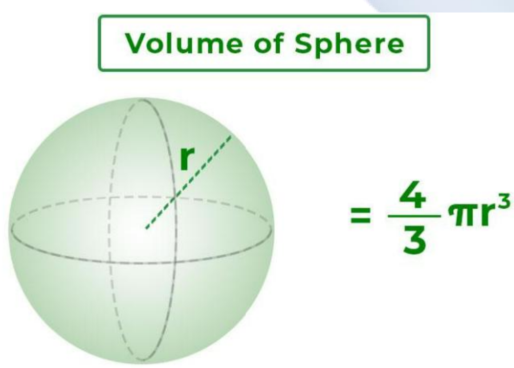
Solution: First find radius:

$$r = d/2 = 21/2 = 10.5 \text{ cm}$$

$$\text{Surface Area} = 4\pi r^2$$
$$= 4 \times (22/7) \times (10.5)^2$$
$$= 4 \times (22/7) \times 110.25$$
$$= 4 \times 22 \times 15.75$$
$$= 1{,}386 \text{ cm}^2$$

Answer: 1,386 cm²

3. Volume of a Sphere



The **volume** is the amount of space occupied by the sphere.

Formula: Volume of Sphere = (4/3)πr³

Example 5: Find the volume of a sphere with radius 3 cm. (Use $\pi = 22/7$)

Solution:

$$\text{Volume} = (4/3)\pi r^3$$
$$= (4/3) \times (22/7) \times 3^3$$
$$= (4/3) \times (22/7) \times 27$$
$$= (4 \times 22 \times 27)/(3 \times 7)$$
$$= 2{,}376/21$$
$$= 113.14 \text{ cm}^3$$

Answer: 113.14 cm³

Example 6: A spherical water tank has radius 1.5 m. Calculate its volume in: a) Cubic meters b) Liters (1 m³ = 1,000 liters)

Solution: a)

$$\text{Volume} = (4/3)\pi r^3$$
$$= (4/3) \times 3.142 \times (1.5)^3$$
$$= (4/3) \times 3.142 \times 3.375$$
$$= 4.189 \times 3.375$$
$$= 14.14 \text{ m}^3$$

b)

$$\text{Volume in liters} = 14.14 \times 1{,}000$$
$$= 14{,}140 \text{ liters}$$

Answer: a) 14.14 m³ b) 14,140 liters

Example 7: The volume of a sphere is 4,851 cm³. Find its radius. (Use $\pi = 22/7$)

Solution:

$$(4/3)\pi r^3 = 4{,}851$$
$$(4/3) \times (22/7) \times r^3 = 4{,}851$$
$$(88/21) \times r^3 = 4{,}851$$
$$r^3 = 4{,}851 \times (21/88)$$
$$r^3 = 4{,}851 \times 21 \div 88$$
$$r^3 = 101{,}871 \div 88$$
$$r^3 = 1{,}157.625$$
$$r = \sqrt[3]{1{,}157.625}$$
$$r \approx 10.5 \text{ cm}$$

Answer: r = 10.5 cm

Example 8: A sphere has diameter 42 cm. Calculate: a) Its surface area b) Its volume

Solution:

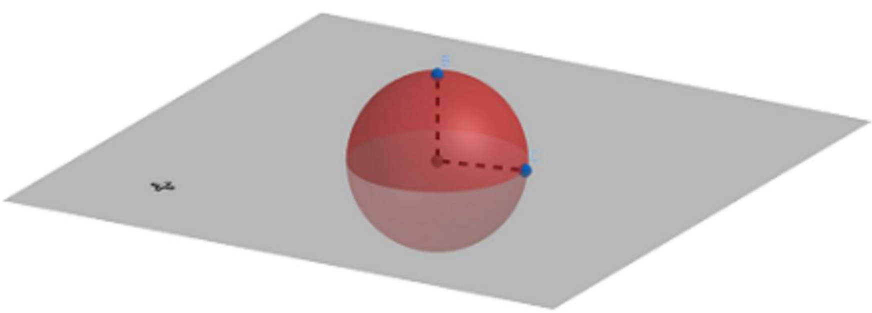
$r = 42/2 = 21$ cm

a) Surface Area $= 4\pi r^2$
$= 4 \times (22/7) \times 21^2$
$= 4 \times (22/7) \times 441$
$= 4 \times 22 \times 63$
$= 5{,}544\ \text{cm}^2$

b) Volume $= (4/3)\pi r^3$
$= (4/3) \times (22/7) \times 21^3$
$= (4/3) \times (22/7) \times 9{,}261$
$= (4 \times 22 \times 9{,}261)/(3 \times 7)$
$= 814{,}968/21$
$= 38{,}808\ \text{cm}^3$

Answer: a) 5,544 cm² b) 38,808 cm³

4. Hemispheres



A **hemisphere** is exactly half of a sphere, divided by a plane passing through its center.

Types of Hemispheres:

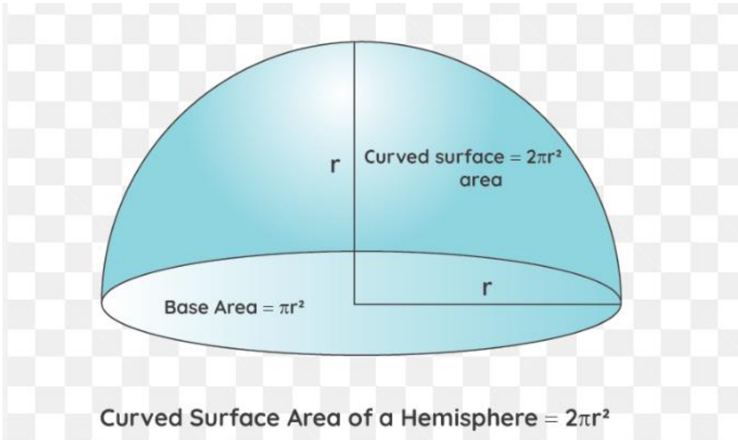
1. **Solid Hemisphere:** The complete half-sphere
2. **Hollow Hemisphere:** Like a bowl (has thickness)

Parts of a Hemisphere:

- **Curved surface:** The spherical part
- **Flat surface (base):** Circular base
- **Radius:** Same as the sphere's radius

5. Curved Surface Area of a Hemisphere

The curved surface area is half the surface area of a sphere.



Formula: Curved Surface Area of Hemisphere = $2\pi r^2$

Example 9: Find the curved surface area of a hemisphere with radius 7 cm.

Solution:

$$\text{Curved Surface Area} = 2\pi r^2$$
$$= 2 \times (22/7) \times 7^2$$
$$= 2 \times (22/7) \times 49$$
$$= 2 \times 22 \times 7$$
$$= 308 \text{ cm}^2$$

Answer: 308 cm^2

Example 10: A hemispherical dome has diameter 28 m. Find its curved surface area.

Solution:

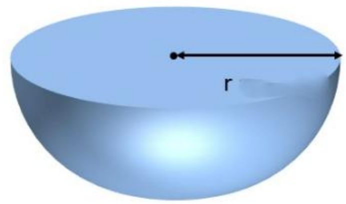
$$r = 28/2 = 14 \text{ m}$$

$$\text{Curved Surface Area} = 2\pi r^2$$
$$= 2 \times (22/7) \times 14^2$$
$$= 2 \times (22/7) \times 196$$
$$= 2 \times 22 \times 28$$
$$= 1{,}232 \text{ m}^2$$

Answer: 1,232 m^2

6. Total Surface Area of a Solid Hemisphere

Total surface area includes both the curved surface and the circular base.



Total surface area of hemisphere = $3\pi r^2$

Formula: Total Surface Area = Curved Surface Area + Area of Base Total Surface Area = $2\pi r^2 + \pi r^2$ Total Surface Area = $3\pi r^2$

Example 11: Find the total surface area of a solid hemisphere with radius 10.5 cm.

Solution:

```
Total Surface Area = 3πr²
= 3 × (22/7) × (10.5)²
= 3 × (22/7) × 110.25
= 3 × 22 × 15.75
= 1,039.5 cm²
```

Answer: 1,039.5 cm²

Example 12: A hemispherical bowl has radius 21 cm. Calculate: a) Curved surface area b) Total surface area

Solution: a)

```
Curved Surface Area = 2πr²
= 2 × (22/7) × 21²
= 2 × (22/7) × 441
= 2 × 22 × 63
= 2,772 cm²
```

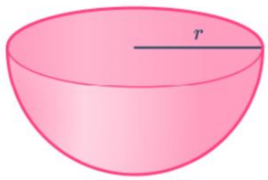
b)

```
Total Surface Area = 3πr²
= 3 × (22/7) × 21²
= 3 × (22/7) × 441
= 3 × 22 × 63
= 4,158 cm²
```

Answer: a) 2,772 cm² b) 4,158 cm²

7. Volume of a Hemisphere

The volume of a hemisphere is half the volume of a sphere.



$$\text{Volume of a hemisphere} = \frac{4}{3}\pi r^3 \div 2$$

or

$$\text{Volume of a hemisphere} = \frac{1}{2} \times \frac{4}{3}\pi r^3 = \frac{2}{3}\pi r^3$$

Formula: Volume of Hemisphere = (2/3)πr³

Example 13: Find the volume of a hemisphere with radius 6 cm.

Solution:

```
Volume = (2/3)πr³
= (2/3) × (22/7) × 6³
= (2/3) × (22/7) × 216
= (2 × 22 × 216)/(3 × 7)
= 9,504/21
= 452.57 cm³
```

Answer: 452.57 cm³

Example 14: A hemispherical tank has diameter 14 m. Find its capacity in liters.

Solution:

```
r = 14/2 = 7 m

Volume = (2/3)πr³
= (2/3) × (22/7) × 7³
= (2/3) × (22/7) × 343
= (2 × 22 × 343)/(3 × 7)
= 15,092/21
= 718.67 m³

Capacity in liters = 718.67 × 1,000
= 718,670 liters
```

Answer: 718,670 liters

8. Hollow Spheres and Hemispheres

Hollow Sphere: A sphere with an empty interior (like a football).

Given:

- External radius = R
- Internal radius = r
- Thickness = R - r

Surface Area of Hollow Sphere:

- External surface = $4\pi R^2$
- Internal surface = $4\pi r^2$
- Total surface area = $4\pi R^2 + 4\pi r^2 = 4\pi(R^2 + r^2)$

Volume of Material in Hollow Sphere: Volume = Volume of outer sphere - Volume of inner sphere Volume = $(4/3)\pi R^3 - (4/3)\pi r^3$ Volume = $(4/3)\pi(R^3 - r^3)$

Example 15: A hollow sphere has external radius 10 cm and internal radius 7 cm. Find: a) Volume of material b) Total surface area

Solution: a)

```
Volume = (4/3)π(R³ - r³)
= (4/3) × (22/7) × (10³ - 7³)
= (4/3) × (22/7) × (1,000 - 343)
= (4/3) × (22/7) × 657
= (4 × 22 × 657)/(3 × 7)
= 57,816/21
= 2,753.14 cm³
```

b)

```
Total Surface Area = 4π(R² + r²)
= 4 × (22/7) × (10² + 7²)
= 4 × (22/7) × (100 + 49)
= 4 × (22/7) × 149
= 4 × 22 × 21.29
= 1,873.36 cm²
```

Answer: a) 2,753.14 cm³ b) 1,873.36 cm²

Example 16: A hollow hemispherical bowl has external diameter 16 cm and thickness 2 cm. Find the volume of material used.

Solution:

```
External radius R = 16/2 = 8 cm
Thickness = 2 cm
Internal radius r = 8 - 2 = 6 cm

Volume = (2/3)π(R³ - r³)
```

$= (2/3) \times (22/7) \times (8^3 - 6^3)$
$= (2/3) \times (22/7) \times (512 - 216)$
$= (2/3) \times (22/7) \times 296$
$= (2 \times 22 \times 296)/(3 \times 7)$
$= 13{,}024/21$
$= 620.19 \text{ cm}^3$

Answer: 620.19 cm³

9. Combined Solids Involving Spheres and Hemispheres

Example 17: A toy consists of a hemisphere surmounted by a cone. The radius of the base is 7 cm and the total height is 20 cm. Find the total surface area. (Ignore the base)

Solution:

Hemisphere radius = 7 cm
Height of cone = 20 - 7 = 13 cm
Radius of cone = 7 cm

For cone, find slant height l:
$l = \sqrt{(h^2 + r^2)}$
$l = \sqrt{(13^2 + 7^2)}$
$l = \sqrt{(169 + 49)}$
$l = \sqrt{218}$
$l = 14.76$ cm

Curved surface area of hemisphere = $2\pi r^2$
$= 2 \times (22/7) \times 7^2$
$= 2 \times 22 \times 7$
$= 308 \text{ cm}^2$

Curved surface area of cone = $\pi r l$
$= (22/7) \times 7 \times 14.76$
$= 22 \times 14.76$
$= 324.72 \text{ cm}^2$

Total surface area = 308 + 324.72
$= 632.72 \text{ cm}^2$

Answer: 632.72 cm²

Example 18: A solid consists of a cylinder with hemispheres at both ends. The radius is 3.5 cm and the length of the cylindrical part is 10 cm. Find: a) Total surface area b) Volume

Solution:

r = 3.5 cm
h = 10 cm

a) Surface area of cylinder = $2\pi rh$ (curved surface only)
$= 2 \times (22/7) \times 3.5 \times 10$
$= 2 \times 22 \times 0.5 \times 10$
$= 220\ cm^2$

Surface area of two hemispheres = $2 \times 2\pi r^2$
$= 4\pi r^2$
$= 4 \times (22/7) \times (3.5)^2$
$= 4 \times (22/7) \times 12.25$
$= 4 \times 22 \times 1.75$
$= 154\ cm^2$

Total surface area = $220 + 154 = 374\ cm^2$

b) Volume of cylinder = $\pi r^2 h$
$= (22/7) \times (3.5)^2 \times 10$
$= (22/7) \times 12.25 \times 10$
$= 22 \times 1.75 \times 10$
$= 385\ cm^3$

Volume of two hemispheres = $2 \times (2/3)\pi r^3$
$= (4/3)\pi r^3$ (volume of one sphere)
$= (4/3) \times (22/7) \times (3.5)^3$
$= (4/3) \times (22/7) \times 42.875$
$= (4 \times 22 \times 42.875)/(3 \times 7)$
$= 3{,}773/21$
$= 179.67\ cm^3$

Total volume = $385 + 179.67 = 564.67\ cm^3$

Answer: a) 374 cm² b) 564.67 cm³

10. Word Problems

Example 19: A spherical water tank has internal diameter 4.2 m. How many liters of water can it hold when full? ($1\ m^3 = 1{,}000$ liters)

Solution:

$r = 4.2/2 = 2.1\ m$

Volume = $(4/3)\pi r^3$
$= (4/3) \times (22/7) \times (2.1)^3$
$= (4/3) \times (22/7) \times 9.261$
$= (4 \times 22 \times 9.261)/(3 \times 7)$
$= 814.968/21$
$= 38.808\ m^3$

$$\text{Capacity} = 38.808 \times 1{,}000$$
$$= 38{,}808 \text{ liters}$$

Answer: 38,808 liters

Example 20: How many lead balls of radius 1 cm can be made from a sphere of radius 8 cm?

Solution:

$$\text{Volume of large sphere} = (4/3)\pi R^3$$
$$= (4/3)\pi(8)^3$$
$$= (4/3)\pi \times 512$$

$$\text{Volume of one small ball} = (4/3)\pi r^3$$
$$= (4/3)\pi(1)^3$$
$$= (4/3)\pi$$

$$\text{Number of balls} = \text{Volume of large sphere} / \text{Volume of one ball}$$
$$= [(4/3)\pi \times 512] / [(4/3)\pi]$$
$$= 512$$

Answer: 512 balls

Example 21: A hemispherical bowl of internal radius 9 cm contains water. This water is to be poured into cylindrical bottles of diameter 3 cm and height 4 cm. How many bottles are needed?

Solution:

$$\text{Volume of hemisphere} = (2/3)\pi r^3$$
$$= (2/3) \times (22/7) \times 9^3$$
$$= (2/3) \times (22/7) \times 729$$
$$= (2 \times 22 \times 729)/(3 \times 7)$$
$$= 32{,}076/21$$
$$= 1{,}527.43 \text{ cm}^3$$

$$\text{Volume of one bottle} = \pi r^2 h$$
$$r = 3/2 = 1.5 \text{ cm}$$
$$= (22/7) \times (1.5)^2 \times 4$$
$$= (22/7) \times 2.25 \times 4$$
$$= (22/7) \times 9$$
$$= 198/7$$
$$= 28.29 \text{ cm}^3$$

$$\text{Number of bottles} = 1{,}527.43 / 28.29$$
$$= 54 \text{ bottles (approximately)}$$

Answer: 54 bottles

Example 22: A metallic sphere of radius 10.5 cm is melted and recast into small cones of radius 3.5 cm and height 3 cm. How many cones are formed?

Solution:

Volume of sphere $= (4/3)\pi r^3$
$= (4/3) \times (22/7) \times (10.5)^3$
$= (4/3) \times (22/7) \times 1{,}157.625$
$= (4 \times 22 \times 1{,}157.625)/(3 \times 7)$
$= 101{,}871/21$
$= 4{,}851 \text{ cm}^3$

Volume of one cone $= (1/3)\pi r^2 h$
$= (1/3) \times (22/7) \times (3.5)^2 \times 3$
$= (1/3) \times (22/7) \times 12.25 \times 3$
$= (1/3) \times (22/7) \times 36.75$
$= (22 \times 36.75)/(3 \times 7)$
$= 808.5/21$
$= 38.5 \text{ cm}^3$

Number of cones $= 4{,}851 / 38.5$
$= 126$ cones

Answer: 126 cones

11. Practical Applications

In Architecture:

- Dome structures (mosques, churches, capitol buildings)
- Planetariums
- Geodesic domes

In Sports:

- Calculating material for footballs, basketballs
- Stadium dome coverage

In Manufacturing:

- Ball bearings
- Storage tanks
- Capsules and tablets

In Nature:

- Planet sizes and volumes

- Soap bubbles
- Fruit calculations (oranges, coconuts)

EVALUATION

1. Find the surface area of a sphere with radius 14 cm.
2. Calculate the volume of a sphere with diameter 12 cm. (Use $\pi = 3.142$)
3. A sphere has surface area 616 cm^2. Find its radius. (Use $\pi = 22/7$)
4. Find the curved surface area of a hemisphere with radius 10.5 cm.
5. Calculate the total surface area of a solid hemisphere with radius 7 cm.
6. Find the volume of a hemisphere with diameter 18 cm.
7. A hollow sphere has external radius 12 cm and internal radius 9 cm. Find the volume of material.
8. How many small balls of radius 2 cm can be made from a sphere of radius 6 cm?
9. A hemispherical bowl has internal diameter 21 cm. Find its capacity in cm^3.
10. A spherical water tank has radius 3.5 m. How many liters of water can it hold?

ASSIGNMENT

1. **Basic Calculations:** a) Find the surface area and volume of a sphere with radius 21 cm. b) A sphere has volume 38,808 cm^3. Find its radius. (Use $\pi = 22/7$) c) Calculate the curved and total surface area of a hemisphere with radius 14 cm. d) Find the volume of a hemisphere with radius 10.5 cm.
2. **Hollow Solids:** a) A hollow sphere has external diameter 20 cm and thickness 3 cm. Find: i) The volume of material used ii) The total surface area

   b) A hemispherical bowl has internal radius 12 cm and thickness 1 cm. Calculate the volume of material.
3. **Word Problems:** a) A spherical balloon is inflated so that its radius increases from 7 cm to 14 cm. How many times does: i) Its surface area increase? ii) Its volume increase?

   b) A solid metal sphere of radius 10 cm is melted and recast into smaller spheres of radius 2 cm. How many small spheres are formed?

   c) A hemispherical tank of internal radius 1.4 m is full of water. The water is emptied into a cylindrical tank of diameter 2.8 m. Find the height of water in the cylinder.

   d) A solid consisting of a cylinder with hemispherical ends has total length 20 cm and radius 3.5 cm. If the length of the cylindrical part is 13 cm, find: i) Total surface area ii) Volume

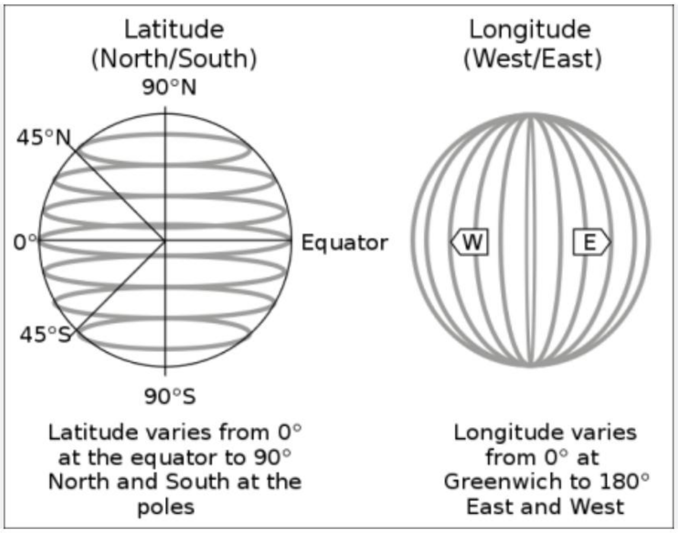
4. **Practical Application:** A company manufactures hemispherical bowls of radius 7 cm. The bowls are made of metal of thickness 0.5 cm. a) Find the external and internal radii b) Calculate the volume of metal in one bowl c) If metal costs ₦50 per cm^3, find the cost of material for one bowl d) If the company adds 40% profit, what is the selling price?

5. **Challenge Problem:** A solid toy is in the form of a hemisphere surmounted by a right circular cone. The height of the cone is 4 cm and the diameter of the base is 8 cm. a) Draw a diagram of the solid b) Find the volume of the toy c) Find the total surface area of the toy d) If the toy is to be painted, and 1 ml of paint covers 10 cm^2, how much paint is needed?

WEEK 9: LONGITUDE AND LATITUDE I

CONTENT

1. Introduction to Earth's Coordinate System



The **Earth** is approximately spherical (actually an oblate spheroid - slightly flattened at the poles). To locate positions on Earth, we use a coordinate system based on **latitude** and **longitude**.

Key Facts:

- Earth's radius ≈ 6,400 km (or 6,370 km - varies slightly)
- One complete rotation = 360°
- One rotation takes 24 hours
- Circumference at equator ≈ 40,000 km

2. Important Terms and Definitions

A. North and South Poles

North Pole:

- The northernmost point on Earth
- Latitude 90°N
- All longitude lines converge here

South Pole:

- The southernmost point on Earth
- Latitude 90°S
- All longitude lines converge here

B. Equator

The **equator** is an imaginary line running around the middle of the Earth, halfway between the North and South Poles.

Properties:

- Latitude 0°
- Divides Earth into Northern and Southern Hemispheres
- Length ≈ 40,000 km
- All points on equator are equidistant from both poles
- Perpendicular to Earth's axis

C. Meridian (Line of Longitude)

A **meridian** is a great circle passing through both poles.

Properties:

- Runs from North Pole to South Pole
- All meridians are equal in length (approximately 40,000 km)
- Measured in degrees East or West of the Prime Meridian
- Also called "lines of longitude"

Prime Meridian (Greenwich Meridian):

- Longitude 0°
- Passes through Greenwich, London, England
- Divides Earth into Eastern and Western Hemispheres
- Reference line for all longitude measurements

D. Latitude (Parallel)

Latitude lines are circles parallel to the equator.

Properties:

- Run East-West
- Range from 0° (equator) to 90°N (North Pole) and 90°S (South Pole)
- All points on the same latitude are the same distance from the equator
- Circles get smaller as you move toward the poles
- Also called "parallels of latitude"

E. Great Circle

A **great circle** is any circle on the sphere's surface whose center coincides with the center of the Earth.

Properties:

- Largest possible circle on a sphere

- Divides sphere into two equal hemispheres
- All meridians are great circles
- Only the equator is a great circle among parallels of latitude
- Shortest distance between two points on Earth follows a great circle

F. Small Circle

Any circle on Earth's surface that is not a great circle.

Properties:

- Center does not coincide with Earth's center
- All parallels of latitude (except equator) are small circles
- Radius less than Earth's radius

3. Understanding Latitude

Notation:

- Northern Hemisphere: Marked as N or + (e.g., 30°N)
- Southern Hemisphere: Marked as S or - (e.g., 30°S)

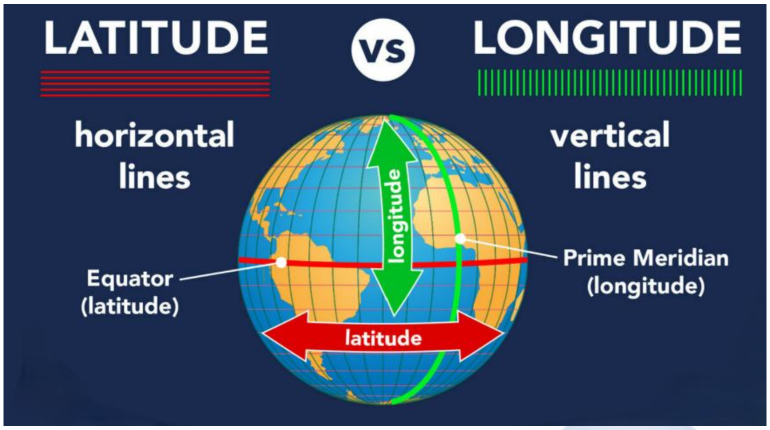
Important Latitudes:

- **Arctic Circle:** 66.5°N
- **Tropic of Cancer:** 23.5°N
- **Equator:** 0°
- **Tropic of Capricorn:** 23.5°S
- **Antarctic Circle:** 66.5°S

Example 1: Lagos, Nigeria is at approximately 6.5°N. What does this mean?

Answer: Lagos is 6.5 degrees north of the equator, in the Northern Hemisphere.

4. Understanding Longitude



Notation:

- Eastern Hemisphere: Marked as E or + (e.g., 40°E)
- Western Hemisphere: Marked as W or - (e.g., 40°W)
- Ranges from 0° to 180°E and 0° to 180°W

International Date Line:

- Approximately follows 180° longitude
- Where date changes
- Located in the Pacific Ocean

Example 2: Abuja, Nigeria is at approximately 7.5°E. What does this mean?

Answer: Abuja is 7.5 degrees east of the Prime Meridian, in the Eastern Hemisphere.

5. Coordinates of a Location

A location on Earth is specified by two coordinates: latitude and longitude.

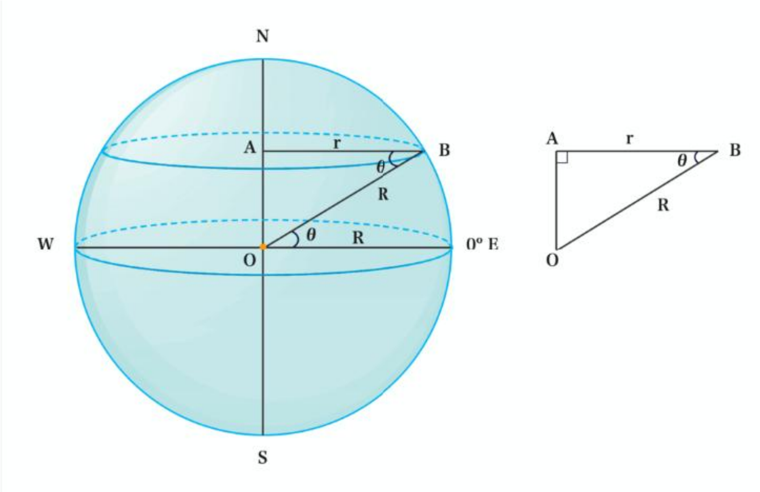
Format: (Latitude, Longitude)

Examples:

- **Lagos, Nigeria:** (6.5°N, 3.4°E)
- **London, UK:** (51.5°N, 0°) - on Prime Meridian
- **New York, USA:** (40.7°N, 74.0°W)
- **Sydney, Australia:** (33.9°S, 151.2°E)

6. Radius of a Parallel of Latitude

The radius of a parallel of latitude decreases as you move away from the equator toward the poles.



Formula: $r = R \cos \theta$

Where:

- **r** = radius of the parallel of latitude
- **R** = radius of Earth (≈ 6,400 km)
- **θ** = angle of latitude

Important Notes:

- At equator ($\theta = 0°$): $r = R \cos 0° = R$ (maximum radius)
- At poles ($\theta = 90°$): $r = R \cos 90° = 0$ (point)
- Radius decreases as latitude increases

Example 3: Find the radius of the parallel of latitude 60°N if Earth's radius is 6,400 km.

Solution:

```
r = R cos θ
r = 6,400 × cos 60°
r = 6,400 × 0.5
r = 3,200 km
```

Answer: 3,200 km

Example 4: Calculate the radius of the circle of latitude 45°S. (R = 6,400 km)

Solution:

```
r = R cos θ
r = 6,400 × cos 45°
r = 6,400 × (√2/2)
r = 6,400 × 0.7071
r = 4,525.44 km
```

Answer: 4,525.44 km

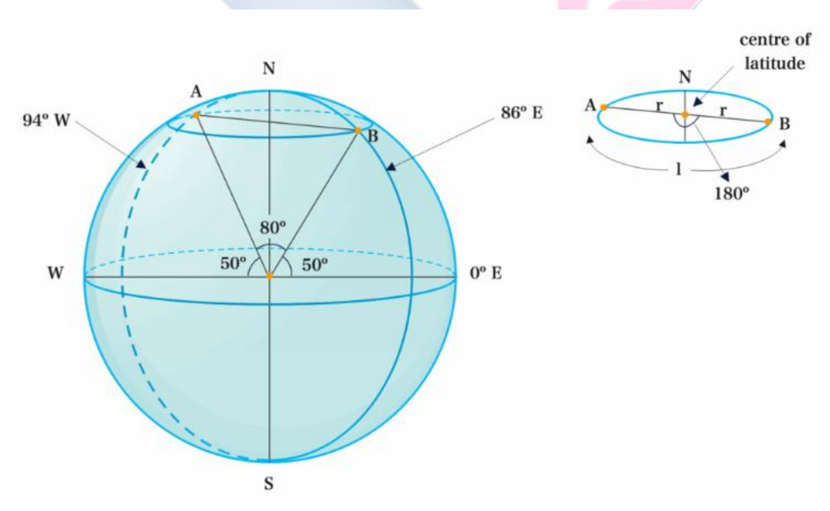
Example 5: The radius of a parallel of latitude is 3,200 km. Find the latitude if Earth's radius is 6,400 km.

Solution:

```
r = R cos θ
3,200 = 6,400 cos θ
cos θ = 3,200/6,400
cos θ = 0.5
θ = cos⁻¹(0.5)
θ = 60°
```

Answer: 60°N or 60°S

7. Length (Circumference) of a Parallel of Latitude



The circumference of a parallel of latitude can be calculated once we know its radius.

Formula: Length = 2πr = 2πR cos θ

Where:

- **R** = Earth's radius
- **θ** = latitude angle

- **r** = radius of that latitude

Example 6: Find the length of the parallel of latitude 60°N. (R = 6,400 km, π = 22/7)

Solution:

```
First find radius:
r = R cos 60° = 6,400 × 0.5 = 3,200 km

Length = 2πr
= 2 × (22/7) × 3,200
= 2 × 22 × 457.14
= 20,114.29 km
```

Answer: 20,114.29 km

Example 7: What is the circumference of Earth at latitude 30°S?

Solution:

```
r = R cos 30°
r = 6,400 × (√3/2)
r = 6,400 × 0.866
r = 5,542.4 km

Circumference = 2πr
= 2 × 3.142 × 5,542.4
= 34,805.83 km
```

Answer: 34,805.83 km

8. Angular Distance Between Two Points

On the Same Meridian (Same Longitude): The angular distance is simply the difference in latitudes.

Example 8: Find the angular distance between points A(30°N, 15°E) and B(10°S, 15°E).

Solution: Both points are on the same meridian (15°E).

```
Angular distance = 30° + 10° = 40°
(Add when on opposite sides of equator)
```

Answer: 40°

Example 9: Find the angular distance between C(50°N, 20°W) and D(20°N, 20°W).

Solution: Both on same meridian (20°W).

```
Angular distance = 50° - 20° = 30°
(Subtract when on same side of equator)
```

Answer: 30°

On the Same Parallel (Same Latitude): The angular distance depends on the difference in longitudes.

Example 10: Two points P and Q are on latitude 60°N. P is on longitude 30°E and Q is on longitude 50°E. Find the angular distance between them along the parallel of latitude.

Solution:

```
Difference in longitude = 50° - 30° = 20°

Note: This is the angle at Earth's center measured along the parallel.
```

Answer: 20°

However, when calculating distances along a parallel (not a great circle), we need to account for the smaller radius.

9. Distance Calculations Along Meridians

Since all meridians are great circles with the same length, distance calculations are straightforward.

Formula: Distance = (θ/360°) × 2πR

Or more simply: **Distance = (θ/360°) × 40,000 km**

Where θ is the angular distance in degrees.

Approximation: 1° along a meridian ≈ 111 km (since 40,000 ÷ 360 ≈ 111)

Example 11: Find the distance between two points on the same meridian if their latitudes are 40°N and 10°N. (Use R = 6,400 km)

Solution:

```
Angular distance = 40° - 10° = 30°

Distance = (θ/360°) × 2πR
= (30/360) × 2 × (22/7) × 6,400
= (1/12) × 2 × (22/7) × 6,400
= (2 × 22 × 6,400)/(12 × 7)
= 281,600/84
= 3,352.38 km
```

Alternative method:

```
Distance = 30° × 111 km/degree
= 3,330 km (approximate)
```

Answer: 3,352.38 km (or approximately 3,330 km)

Example 12: Two towns A and B are on longitude 45°E. A is on latitude 60°N and B is on latitude 30°S. Find the distance between them in kilometers.

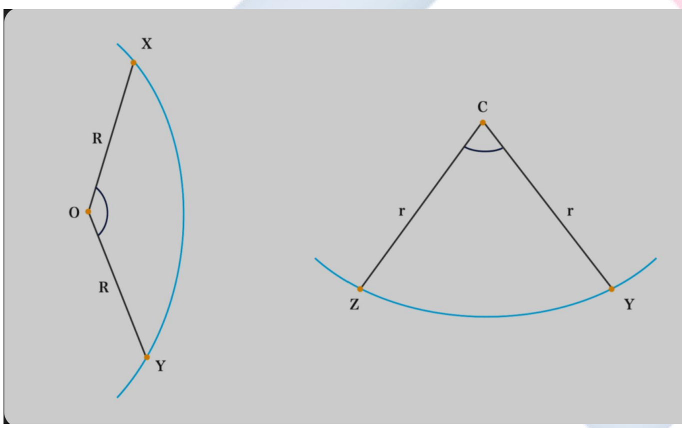
Solution:

```
Angular distance = 60° + 30° = 90°
(Add because on opposite sides of equator)

Distance = (90/360) × 2πR
= (1/4) × 2 × (22/7) × 6,400
= (1/4) × 40,914.29
= 10,228.57 km
```

Answer: 10,228.57 km

10. Arc Length Along a Parallel of Latitude



For distances along a parallel of latitude (not along a meridian), we use the radius of that particular latitude.

Formula: Distance = (θ/360°) × 2πr

Where:

- θ = difference in longitude
- r = R cos(latitude)

Example 13: Two points P and Q are both on latitude 60°N. P is on longitude 20°E and Q is on longitude 80°E. Find the distance between them measured along the parallel of latitude.

Solution:

```
Latitude = 60°N
Difference in longitude = 80° - 20° = 60°

First find radius of latitude 60°N:
r = R cos 60°
r = 6,400 × 0.5
r = 3,200 km

Distance along parallel = (θ/360°) × 2πr
= (60/360) × 2 × (22/7) × 3,200
= (1/6) × 2 × (22/7) × 3,200
= (2 × 22 × 3,200)/(6 × 7)
= 140,800/42
= 3,352.38 km
```

Answer: 3,352.38 km

Example 14: Two ships A and B are on latitude 45°S. A is on longitude 30°W and B is on longitude 45°W. Find the distance between them along the latitude circle.

Solution:

```
Difference in longitude = 45° - 30° = 15°

r = R cos 45°
r = 6,400 × 0.7071
r = 4,525.44 km

Distance = (15/360) × 2 × π × 4,525.44
= (15/360) × 2 × 3.142 × 4,525.44
= 0.04167 × 28,418.06
= 1,184.08 km
```

Answer: 1,184.08 km

11. Relationship: Distance Along Parallel vs. Along Meridian

For the same angular distance: **Distance along parallel = (Distance along meridian) × cos(latitude)**

Example 15: The distance between two points on the same meridian is 4,000 km. If they were on the same parallel at latitude 60°N, what would be the distance for the same angular difference?

Solution:

```
Distance along parallel = Distance along meridian × cos(latitude)
= 4,000 × cos 60°
```

$= 4,000 \times 0.5$
$= 2,000$ km

Answer: 2,000 km

12. Finding Positions Given Distances

Example 16: A point P is at latitude 40°N, longitude 10°E. Another point Q is 3,330 km due north of P along the same meridian. Find the coordinates of Q.

Solution:

Distance along meridian = 3,330 km

Angular distance = $3,330/111 \approx 30°$

Since Q is north of P:
Latitude of Q = 40° + 30° = 70°N
Longitude of Q = 10°E (same meridian)

Answer: Q(70°N, 10°E)

Example 17: Two places X and Y are on latitude 60°N. The distance between them along the parallel is 2,000 km. If X is on longitude 20°E, find the longitude of Y (to the east of X).

Solution:

$r = R \cos 60° = 6,400 \times 0.5 = 3,200$ km

Circumference of parallel = $2\pi r$
$= 2 \times 3.142 \times 3,200$
$= 20,108.8$ km

Distance corresponds to angle:
$\theta = (\text{Distance}/\text{Circumference}) \times 360°$
$\theta = (2,000/20,108.8) \times 360°$
$\theta = 0.0995 \times 360°$
$\theta = 35.8°$

Longitude of Y = 20° + 35.8° = 55.8°E

Answer: 55.8°E

13. Practical Applications

Navigation:

- Ships and aircraft use latitude and longitude for navigation

- GPS systems use these coordinates
- Plotting routes and calculating distances

Geography:

- Mapping locations
- Climate zones (based on latitude)
- Time zones (based on longitude)

Communication:

- Satellite positioning
- Weather forecasting
- Emergency location services

EVALUATION

1. Define the following terms: a) Equator b) Meridian c) Great circle d) Parallel of latitude
2. What are the coordinates of the North Pole?
3. Calculate the radius of latitude 30°N if Earth's radius is 6,400 km.
4. Find the circumference of the parallel of latitude 45°S. (R = 6,400 km)
5. Two points are on the same meridian at latitudes 50°N and 20°N. Find the angular distance between them.
6. Find the distance between points A(60°N, 30°E) and B(20°N, 30°E) in kilometers.
7. Two towns P and Q are on latitude 60°N. P is at 40°E and Q is at 70°E. Find the distance between them along the parallel.
8. The radius of a parallel of latitude is 4,800 km. Find the latitude. (R = 6,400 km)
9. What is the approximate distance represented by 1° along a meridian?
10. Differentiate between a great circle and a small circle.

ASSIGNMENT

1. **Basic Concepts:** a) Explain why all meridians are great circles but only the equator is a great circle among parallels. b) Draw a diagram showing the equator, Prime Meridian, and a point at (30°N, 40°E). c) State the coordinates of Lagos (approximately 6.5°N, 3.4°E) and explain what each coordinate means.
2. **Radius and Circumference Calculations:** a) Find the radius of the following parallels of latitude: (R = 6,400 km) i) 0° (Equator) ii) 30°N iii) 60°S iv) 90°N

b) Calculate the circumference of latitude 50°N.

c) At what latitude is the radius of the parallel exactly half of Earth's radius?

3. **Distance Problems:** a) Find the distance between: i) (30°N, 20°E) and (50°N, 20°E) ii) (40°N, 15°W) and (10°S, 15°W)

   b) Two cities A and B are on the same meridian. A is at 55°N and the distance between them is 5,550 km. Find the possible latitudes of B.

   c) Two points X and Y are on latitude 60°N at longitudes 10°W and 35°E respectively. Calculate: i) The difference in longitude ii) The distance between them along the parallel

4. **Application Problems:** a) A ship sails from point P(20°N, 40°E) due north for 4,440 km. Find its new position.

   b) An aircraft flies from A(60°N, 30°W) to B(60°N, 20°E) along the parallel of latitude. Calculate the distance covered.

   c) Two weather stations are 3,200 km apart on the same parallel of latitude at 45°S. If one station is at longitude 120°E, find the longitude of the other station.

5. **Challenge Problem:** Three towns A, B, and C are located as follows:
   - A is at (0°, 30°E)
   - B is at (60°N, 30°E)
   - C is at (60°N, 60°E)

   Calculate: a) The distance from A to B b) The distance from B to C c) If you travel from A to C, which route is shorter: A→B→C or directly from A to C along a great circle? (Hint: For the direct route, you'll need to consider great circle distance, which is more complex)

WEEK 10: LONGITUDE AND LATITUDE II

CONTENT

1. Distance Between Two Points on Earth

We've learned how to calculate distances when points are on:

- The same meridian (same longitude)
- The same parallel (same latitude)

Now we'll look at more complex situations.

Summary of Methods:

Case 1: Same Meridian (Same Longitude)

```
Distance = (Difference in latitude/360°) × 2πR
Distance ≈ (Difference in latitude) × 111 km
```

Case 2: Same Parallel (Same Latitude)

```
r = R cos(latitude)
Distance = (Difference in longitude/360°) × 2πr
```

Case 3: Different Meridian and Latitude (Great Circle Distance) Uses more complex formulas (spherical trigonometry) - typically covered in advanced courses.

2. Detailed Examples of Distance Calculations

Example 1: Calculate the distance between Cape Town (34°S, 18°E) and Nairobi (1°S, 37°E) if both cities were on the same meridian. (R = 6,400 km)

Solution: Note: For this problem, we're asked to calculate as if they were on the same meridian, even though they're not in reality.

```
Difference in latitude = 34° - 1° = 33°

Distance = (33/360) × 2πR
= (33/360) × 2 × (22/7) × 6,400
= (33/360) × 40,228.57
= 0.0917 × 40,228.57
= 3,688 km
```

Answer: 3,688 km

Example 2: Two places P and Q are on latitude 50°N. P is at longitude 80°W and Q is at longitude 30°E. Find the distance between them: a) Along the parallel of latitude b) If they were on the equator with the same longitude difference

Solution:

a) Along parallel at 50°N:

```
Difference in longitude = 80° + 30° = 110°
(Add when one is West and other is East)

r = R cos 50°
r = 6,400 × 0.643
r = 4,115.2 km

Distance = (110/360) × 2πr
= (110/360) × 2 × 3.142 × 4,115.2
= 0.3056 × 25,876.9
= 7,908 km
```

b) Along equator (if same longitude difference):

```
R_equator = 6,400 km (no cosine factor)

Distance = (110/360) × 2π × 6,400
= 0.3056 × 40,212.8
= 12,289 km
```

Answer: a) 7,908 km b) 12,289 km

Example 3: Town A is at (35°N, 40°E) and town B is at (35°N, 70°W). Find the shorter distance between them along the parallel of latitude.

Solution:

```
Both are at 35°N but one is East and one is West.

Longitude difference (going east/west):
Option 1: 40° + 70° = 110° (going westward from A)
Option 2: 360° - 110° = 250° (going eastward from A)

Shorter route = 110°

r = R cos 35°
r = 6,400 × 0.819
r = 5,241.6 km

Distance = (110/360) × 2πr
= (110/360) × 2 × 3.142 × 5,241.6
= 0.3056 × 32,917.89
= 10,060 km
```

Answer: 10,060 km

3. Introduction to Time Calculations

The Earth rotates 360° in 24 hours.

Key Relationships:

- **360° = 24 hours**
- **15° = 1 hour** (since 360 ÷ 24 = 15)
- **1° = 4 minutes** (since 60 ÷ 15 = 4)
- **1' (1 minute of arc) = 4 seconds of time**

Direction of Rotation: Earth rotates from **West to East**

Consequences:

- Places to the **East** experience sunrise/noon/sunset **earlier**
- Places to the **West** experience these events **later**
- When it's noon at one location, places to the east have already passed noon (afternoon), while places to the west haven't reached noon yet (morning)

4. Time Difference Between Two Places

Formula: Time difference = (Difference in longitude) × 4 minutes

Or: **Time difference = (Difference in longitude) ÷ 15 hours**

Important Rule:

- If place B is **East** of place A, time at B is **ahead** (later)
- If place B is **West** of place A, time at B is **behind** (earlier)

Example 4: Find the time difference between two places on longitudes 45°E and 75°E.

Solution:

```
Difference in longitude = 75° - 45° = 30°

Time difference = 30° × 4 minutes
= 120 minutes
= 2 hours
```

Answer: 2 hours (75°E is 2 hours ahead of 45°E)

Example 5: If it is 2:00 PM at longitude 30°E, what is the time at longitude 60°E?

Solution:

```
Difference = 60° - 30° = 30°
```

```
Time difference = 30° × 4 = 120 minutes = 2 hours

Since 60°E is east of 30°E, it is ahead.

Time at 60°E = 2:00 PM + 2 hours = 4:00 PM
```

Answer: 4:00 PM

Example 6: When it is 9:00 AM Monday at 15°W, what is the time and day at 45°E?

Solution:

```
Difference = 15° + 45° = 60°
(Add when one is West and other is East)

Time difference = 60° × 4 = 240 minutes = 4 hours

45°E is east of 15°W, so time is ahead.

Time at 45°E = 9:00 AM + 4 hours = 1:00 PM Monday
```

Answer: 1:00 PM Monday

Example 7: If it is 6:30 PM Tuesday at longitude 120°E, find the time and day at longitude 30°W.

Solution:

```
Difference = 120° + 30° = 150°

Time difference = 150° × 4 = 600 minutes = 10 hours

30°W is west of 120°E, so time is behind.

Time at 30°W = 6:30 PM - 10 hours = 8:30 AM Tuesday
```

Answer: 8:30 AM Tuesday

5. Greenwich Mean Time (GMT)

Greenwich Mean Time (GMT) or **Universal Time Coordinated (UTC)** is the time at the Prime Meridian (0° longitude) in Greenwich, London.

Purpose:

- International time standard

- Reference for all time zones
- Used in aviation, shipping, and international communication

Time Zones: The world is divided into time zones, each generally 15° of longitude wide.

Example 8: If GMT is 10:00 AM, what is the local time at: a) 30°E b) 75°W

Solution:

a) At 30°E:

```
Time difference = 30° × 4 = 120 minutes = 2 hours
30°E is ahead of GMT.
Local time = 10:00 AM + 2 hours = 12:00 noon
```

b) At 75°W:

```
Time difference = 75° × 4 = 300 minutes = 5 hours
75°W is behind GMT.
Local time = 10:00 AM - 5 hours = 5:00 AM
```

Answer: a) 12:00 noon b) 5:00 AM

Example 9: A plane leaves London (0°) at 8:00 AM GMT and flies to Lagos (15°E). If the flight takes 6 hours, at what local time does it arrive in Lagos?

Solution:

```
Arrival time in GMT = 8:00 AM + 6 hours = 2:00 PM GMT

Time difference between Lagos and London:
15° × 4 = 60 minutes = 1 hour

Lagos is ahead of GMT by 1 hour.

Local arrival time in Lagos = 2:00 PM + 1 hour = 3:00 PM
```

Answer: 3:00 PM

6. International Date Line

The **International Date Line** is an imaginary line approximately along longitude 180°, where the date changes.

Properties:

- Located in the Pacific Ocean

- Deviates to avoid land masses
- When crossing from **West to East**, you **subtract one day**
- When crossing from **East to West**, you **add one day**

Example 10: A ship crosses the International Date Line from east to west on Monday at 3:00 PM. What is the day and time immediately after crossing?

Solution:

```
Crossing from east to west: Add one day

Day and time after crossing: Tuesday, 3:00 PM
```

Answer: Tuesday, 3:00 PM

7. Nautical Miles

A **nautical mile** is a unit of distance used in navigation.

Definition:

- **1 nautical mile** = distance corresponding to **1 minute of arc** (1') along a great circle
- 1 nautical mile ≈ 1.852 km
- 1° = 60 nautical miles (since 1° = 60')

Relationship: Along a meridian:

- **Distance in nautical miles = Angular distance in minutes**

Example 11: Two ships are on the same meridian at latitudes 35°N and 37°30'N. Find the distance between them in: a) Nautical miles b) Kilometers

Solution:

```
Angular distance = 37°30' - 35°00' = 2°30'
Convert to minutes: 2°30' = (2 × 60) + 30 = 150'

a) Distance = 150 nautical miles

b) Distance in km = 150 × 1.852
   = 277.8 km
```

Answer: a) 150 nautical miles b) 277.8 km

Example 12: How many nautical miles is 1° 15' 30"?

Solution:

```
Convert to minutes:
1° = 60'
15' = 15'
30" = 30/60 = 0.5'

Total = 60 + 15 + 0.5 = 75.5'

Distance = 75.5 nautical miles
```

Answer: 75.5 nautical miles

8. Speed Calculations

Knots: Speed in nautical miles per hour.

- **1 knot = 1 nautical mile per hour**

Example 13: A ship travels 240 nautical miles in 6 hours. Find its speed in: a) Knots b) Kilometers per hour

Solution:

```
a) Speed = Distance/Time
   = 240/6
   = 40 knots

b) Speed in km/h = 40 × 1.852
   = 74.08 km/h
```

Answer: a) 40 knots b) 74.08 km/h

Example 14: An aircraft flies from P(40°N, 20°E) to Q(40°N, 50°E) at a speed of 500 km/h. How long does the journey take?

Solution:

```
Difference in longitude = 50° - 20° = 30°

r = R cos 40°
r = 6,400 × 0.766
r = 4,902.4 km

Distance = (30/360) × 2πr
= (30/360) × 2 × 3.142 × 4,902.4
= 0.0833 × 30,796.5
= 2,566 km

Time = Distance/Speed
```

```
= 2,566/500
= 5.13 hours
= 5 hours 8 minutes
```

Answer: 5 hours 8 minutes

9. Combined Problems

Example 15: A ship leaves port A(60°N, 40°W) and sails due south to port B on the equator. It then sails due east along the equator to port C at longitude 20°E. Calculate: a) Distance from A to B b) Distance from B to C c) Total distance

Solution:

a) From A to B (along meridian from 60°N to 0°):

```
Angular distance = 60°

Distance = 60° × 111 km
= 6,660 km
```

b) From B to C (along equator):

```
B is at (0°, 40°W)
C is at (0°, 20°E)

Difference = 40° + 20° = 60°

Distance = (60/360) × 2πR
= (60/360) × 40,212.8
= 6,702 km
```

c) Total distance:

```
Total = 6,660 + 6,702 = 13,362 km
```

Answer: a) 6,660 km b) 6,702 km c) 13,362 km

Example 16: An aircraft leaves Lagos (6°30'N, 3°30'E) at 6:00 AM local time on Monday and flies for 5 hours to London (51°30'N, 0°). Calculate: a) The time difference between Lagos and London b) The arrival time in London (local time) c) The GMT at departure

Solution:

a) Longitude difference = 3°30' - 0° = 3°30'

```
Convert to minutes: 3.5° × 4 = 14 minutes

Lagos is ahead of London by 14 minutes.
```

b) Arrival time:

```
Departure (Lagos time): 6:00 AM Monday
Flight time: 5 hours
Arrival (Lagos time): 11:00 AM Monday

London is 14 minutes behind Lagos.
Arrival (London time) = 11:00 AM - 14 minutes
= 10:46 AM Monday
```

c) GMT at departure:

```
Lagos time: 6:00 AM
GMT = 6:00 AM - 14 minutes = 5:46 AM
```

Answer: a) 14 minutes b) 10:46 AM Monday c) 5:46 AM GMT

10. Practical Applications

Aviation:

- Flight planning and scheduling
- Time zone calculations for arrivals
- Fuel calculations based on distances

Shipping:

- Navigation using nautical miles
- Route planning
- Arrival time predictions

Communication:

- Scheduling international meetings
- Broadcasting times
- Satellite communication

Tourism:

- Understanding jet lag
- Planning itineraries across time zones
- Hotel bookings and check-in times

EVALUATION

1. Find the time difference between places at 45°E and 90°E.
2. If it is 3:00 PM at 60°W, what is the time at 30°E?

3. When GMT is 12:00 noon, what is the local time at 75°E?

4. Two ships are on the same meridian at latitudes 30°15'N and 28°45'N. Find the distance between them in nautical miles.

5. Convert 120 nautical miles to kilometers.

6. A plane flies from A(20°N, 40°E) to B(20°N, 70°E) at 600 km/h. How long does the journey take?

7. What is the time at 150°W when it is 6:00 PM Monday at 30°E?

8. Find the distance along the equator between longitudes 20°W and 50°E in kilometers.

9. A ship travels at 25 knots for 8 hours. How far does it travel in kilometers?

10. If you cross the International Date Line from west to east on Friday at 2:00 PM, what is the day and time after crossing?

ASSIGNMENT

1. **Time Calculations:** a) Find the time difference between the following pairs of longitudes: i) 15°E and 45°E ii) 30°W and 60°E iii) 120°E and 150°W

   b) If it is 8:30 AM Wednesday at 75°W, find the time and day at: i) 45°E ii) 165°W iii) 0° (GMT)

2. **Distance Problems:** a) Calculate the distance in both kilometers and nautical miles between: i) (25°30'N, 10°E) and (32°45'N, 10°E) ii) Points on the equator at 40°W and 20°E

   b) Two places X and Y are on latitude 45°N. X is at 30°W and Y is at 15°E. Find: i) The distance between them along the parallel ii) The time difference iii) If it's noon at X, what time is it at Y?

3. **Speed and Time:** a) A ship sails from P(0°, 20°W) due east along the equator at 30 knots. How long will it take to reach Q(0°, 40°E)?

   b) An aircraft leaves city A(30°N, 45°E) at 10:00 AM local time and arrives at city B(30°N, 90°E) at 2:00 PM local time. Calculate: i) The time difference between the cities ii) The actual flight time iii) The distance covered iv) The average speed of the aircraft

4. **GMT Problems:** a) A conference call is scheduled for 3:00 PM GMT. What is the local time for participants in: i) Lagos (15°E) ii) New York (75°W) iii) Tokyo (135°E)

   b) A football match starts at 8:00 PM in London (0°). At what local time can fans watch it live in: i) Abuja (7.5°E) ii) Los Angeles (120°W)

5. **Complex Navigation Problem:** A ship leaves port X at (60°N, 30°W) at 6:00 AM local time on Monday. It sails:
   - First leg: Due south to the equator (takes 2 days at 15 knots)
   - Second leg: Due east along the equator for 4,000 km (at 20 knots)

- Third leg: Due north to latitude 45°N

Calculate: a) The position after the first leg b) Time taken for the second leg c) The position after the second leg d) Distance for the third leg e) Final position f) The day and local time of arrival at the final destination g) The total journey time

---

WEEK 11: BINARY OPERATIONS

CONTENT

1. Definition of Binary Operation

A **binary operation** is a rule that combines any two elements from a set to produce another element (usually from the same set).

Notation: Usually denoted by symbols like *, ∗ , ⊕, ⊗ , #, △, etc.

General Form: If * is a binary operation on set S, then for any a, b ∈ S: **a * b = result**

Examples of Common Binary Operations:

- **Addition (+):** $a + b$
- **Subtraction (−):** $a - b$
- **Multiplication (×):** $a \times b$
- **Division (÷):** $a \div b$ (not defined for $b = 0$)

Example 1: Define a binary operation * on the set of integers by: **a * b = a + 2b**

Evaluate: a) 3 * 4 b) 5 * 2

Solution:

```
a) 3 * 4 = 3 + 2(4) = 3 + 8 = 11

b) 5 * 2 = 5 + 2(2) = 5 + 4 = 9
```

Answer: a) 11 b) 9

Example 2: If $a \oplus b = 2a - b$, find: a) $6 \oplus 3$ b) $(4 \oplus 2) \oplus 5$

Solution:

```
a) 6 ⊕ 3 = 2(6) - 3 = 12 - 3 = 9

b) First find 4 ⊕ 2:
   4 ⊕ 2 = 2(4) - 2 = 8 - 2 = 6

   Then find 6 ⊕ 5:
   6 ⊕ 5 = 2(6) - 5 = 12 - 5 = 7
```

Answer: a) 9 b) 7

2. Properties of Binary Operations

A. Closure Property

A set S is **closed** under operation * if: For all a, b ∈ S, then a * b ∈ S

Example 3: Check if the set of natural numbers N = {1, 2, 3, ...} is closed under: a) Addition b) Subtraction

Solution:

```
a) Addition:
   For any natural numbers a and b, a + b is also a natural number.
   Example: 3 + 5 = 8 ∈ N
   ✓ Closed under addition

b) Subtraction:
   5 - 8 = -3 ∉ N (not a natural number)
   ✗ Not closed under subtraction
```

Example 4: Is the operation $a * b = a^2 + b^2$ closed on the set {0, 1, 2}?

Solution: Check all possible combinations:

```
0 * 0 = 0² + 0² = 0 ✓
0 * 1 = 0² + 1² = 1 ✓
0 * 2 = 0² + 2² = 4 ✗ (4 is not in the set)
```

Since 0 * 2 = 4 ∉ {0, 1, 2}, **not closed**.

B. Commutative Property

An operation * is **commutative** if: **a * b = b * a** for all a, b in the set

Example 5: Check if the operation a * b = a + 2b is commutative.

Solution:

```
Test with a = 3, b = 4:

a * b = 3 + 2(4) = 11
b * a = 4 + 2(3) = 10

Since 11 ≠ 10, not commutative.
```

Example 6: Is multiplication commutative on real numbers?

Solution:

```
For any real numbers a and b:
a × b = b × a
```

```
Example: 3 × 5 = 15 and 5 × 3 = 15

Yes, multiplication is commutative.
```

C. Associative Property

An operation * is **associative** if: **(a * b) * c = a * (b * c)** for all a, b, c in the set

Example 7: Check if a * b = a + b + 1 is associative.

Solution:

```
Test with a = 2, b = 3, c = 4:

(a * b) * c:
First: 2 * 3 = 2 + 3 + 1 = 6
Then: 6 * 4 = 6 + 4 + 1 = 11

a * (b * c):
First: 3 * 4 = 3 + 4 + 1 = 8
Then: 2 * 8 = 2 + 8 + 1 = 11

Since both equal 11, this suggests it's associative.
```

To prove it's associative generally:

```
(a * b) * c = (a + b + 1) * c
            = (a + b + 1) + c + 1
            = a + b + c + 2

a * (b * c) = a * (b + c + 1)
            = a + (b + c + 1) + 1
            = a + b + c + 2

They're equal, so it's associative. ✓
```

Example 8: Is subtraction associative?

Solution:

```
Test with a = 10, b = 5, c = 2:

(10 - 5) - 2 = 5 - 2 = 3
10 - (5 - 2) = 10 - 3 = 7

Since 3 ≠ 7, subtraction is not associative.
```

D. Distributive Property

Operation * **distributes over** operation ⊕ if: **a * (b ⊕ c) = (a * b) ⊕ (a * c)**

Example 9: Does multiplication distribute over addition?

Solution:

```
For any numbers a, b, c:

a × (b + c) = ab + ac

Example: 3 × (4 + 5) = 3 × 9 = 27
         (3 × 4) + (3 × 5) = 12 + 15 = 27 ✓

Yes, multiplication distributes over addition.
```

E. Identity Element

An element **e** is an **identity element** for operation * if: **a * e = e * a = a** for all a in the set

Example 10: Find the identity element for: a) Addition on real numbers b) Multiplication on real numbers

Solution:

```
a) For addition:
   a + e = a
   Therefore e = 0

   Identity element for addition is 0.

b) For multiplication:
   a × e = a
   Therefore e = 1

   Identity element for multiplication is 1.
```

Example 11: Find the identity element for a * b = a + b − 3.

Solution:

```
We need: a * e = a

a + e - 3 = a
e - 3 = 0
```

```
e = 3

Check: a * 3 = a + 3 − 3 = a ✓
       3 * a = 3 + a − 3 = a ✓

Identity element is 3.
```

F. Inverse Element

For an element **a**, its **inverse** (denoted a^{-1}) satisfies: **$a * a^{-1} = a^{-1} * a = e$** (where e is the identity element)

Example 12: Find the inverse of 5 under: a) Addition (identity = 0) b) Multiplication (identity = 1)

Solution:

```
a) Under addition:
   5 + x = 0
   x = −5

   Inverse of 5 is −5.

b) Under multiplication:
   5 × x = 1
   x = 1/5

   Inverse of 5 is 1/5.
```

Example 13: For the operation a * b = a + b + 2, identity element is −2. Find the inverse of 5.

Solution:

```
We need: 5 * x = −2

5 + x + 2 = −2
x + 7 = −2
x = −9

Check: 5 * (−9) = 5 + (−9) + 2 = −2 ✓

Inverse of 5 is −9.
```

3. Creating Operation Tables

An **operation table** shows the results of applying a binary operation to all pairs of elements in a finite set.

Example 14: Construct the operation table for a * b = (a + b) mod 4 on the set {0, 1, 2, 3}.

Solution:

```
*  | 0   1   2   3
---|---------------
0  | 0   1   2   3
1  | 1   2   3   0
2  | 2   3   0   1
3  | 3   0   1   2
```

Observations:

- Closure: All results are in {0, 1, 2, 3} ✓
- Identity: 0 is the identity element (first row and column) ✓
- Every element has an inverse ✓
- Commutative: Table is symmetric about main diagonal ✓

Example 15: For the set {1, 2, 3, 4} with operation a * b = LCM(a, b), construct the table.

Solution:

```
*  | 1   2   3   4
---|---------------
1  | 1   2   3   4
2  | 2   2   6   4
3  | 3   6   3  12
4  | 4   4  12   4
```

Observations:

- Not closed (6 and 12 are not in the set) ✗
- Identity: 1 is the identity ✓
- Commutative: symmetric ✓

4. Solving Equations with Binary Operations

Example 16: If a * b = 2a + b, solve for x in: x * 5 = 17

Solution:

```
x * 5 = 17
2x + 5 = 17
2x = 12
x = 6
```

Answer: x = 6

Example 17: Given a ⊕ b = a² − b, solve: a) x ⊕ 3 = 13 b) 2 ⊕ y = −2

Solution:

```
a) x ⊕ 3 = 13
   x² - 3 = 13
   x² = 16
   x = ±4

b) 2 ⊕ y = -2
   2² - y = -2
   4 - y = -2
   y = 6
```

Answer: a) x = ±4 b) y = 6

Example 18: If p * q = p + q + pq, find x such that x * 2 = 10.

Solution:

```
x * 2 = 10
x + 2 + 2x = 10
3x + 2 = 10
3x = 8
x = 8/3
```

Answer: x = 8/3 or 2⅔

5. Composite Operations

Example 19: If a * b = a + b and a ⊕ b = a − b, find: a) (3 * 4) ⊕ 2 b) 5 * (6 ⊕ 2)

Solution:

```
a) First find 3 * 4:
   3 * 4 = 3 + 4 = 7

   Then: 7 ⊕ 2 = 7 - 2 = 5

b) First find 6 ⊕ 2:
   6 ⊕ 2 = 6 - 2 = 4

   Then: 5 * 4 = 5 + 4 = 9
```

Answer: a) 5 b) 9

Example 20: Define a * b = 2a − b and a # b = ab. Evaluate: (4 * 3) # (2 * 1)

Solution:

```
First find 4 * 3:
4 * 3 = 2(4) - 3 = 8 - 3 = 5

Then find 2 * 1:
2 * 1 = 2(2) - 1 = 4 - 1 = 3

Finally: 5 # 3 = 5 × 3 = 15
```

Answer: 15

6. Real-Life Applications

Computer Science:

- Boolean operations (AND, OR, XOR)
- Set operations (union, intersection)
- Database queries

Cryptography:

- Modular arithmetic
- Encryption algorithms

Digital Electronics:

- Logic gates
- Binary arithmetic

Example 21: In digital logic, the XOR (exclusive OR) operation is defined by:

```
a XOR b | 0   1
--------|-------
   0    | 0   1
   1    | 1   0
```

Properties:

- Commutative: a XOR b = b XOR a ✓
- Associative: (a XOR b) XOR c = a XOR (b XOR c) ✓
- Identity: 0 (since a XOR 0 = a) ✓
- Every element is its own inverse: a XOR a = 0 ✓

7. Proving Properties Algebraically

Example 22: Prove that the operation a * b = a + b − 1 is commutative.

Proof:

```
We need to show: a * b = b * a

LHS: a * b = a + b − 1

RHS: b * a = b + a − 1
           = a + b − 1  (by commutativity of addition)

Since LHS = RHS, the operation is commutative. ✓
```

Example 23: Prove that a * b = √(ab) is associative on positive real numbers.

Proof:

```
We need: (a * b) * c = a * (b * c)

LHS: (a * b) * c = (√(ab)) * c
                 = √(√(ab) × c)
                 = √(c√(ab))
                 = (abc)^(1/4) × c^(1/4)
                 = (ab)^(1/4) × c^(1/2)

Actually, let me recalculate:

LHS: (√(ab)) * c = √(√(ab) × c)

Let's use a different approach:
(a * b) * c = √(ab) * c = √(√(ab) · c) = (abc)^(1/4) · c^(1/4) = (abc²)^(1/4)

RHS: a * (b * c) = a * √(bc) = √(a · √(bc)) = (a²bc)^(1/4)

These are not equal in general, so NOT associative. ✗
```

EVALUATION

1. If $a * b = a + 2b - 1$, find: a) $3 * 4$ b) $5 * 2$
2. Given $p \oplus q = p^2 + q$, evaluate $(2 \oplus 3) \oplus 1$.
3. Check if the operation $a * b = ab - a - b$ is commutative.
4. Find the identity element for $a * b = a + b + 3$.

5. If a * b = 2a − b and the identity element is 0, find the inverse of 4.

6. Solve for x: x * 3 = 11, where a * b = 2a + b.

7. Is the set {0, 1, 2} closed under a * b = a + b?

8. Check if a * b = a + b − ab is associative using a = 1, b = 2, c = 3.

9. Construct an operation table for a * b = max(a, b) on {1, 2, 3}.

10. If a # b = ab + a + b, find x such that 2 # x = 14.

ASSIGNMENT

1. **Basic Operations:** a) Define a * b = 3a − 2b. Evaluate: i) 4 * 5 ii) (6 * 2) * 3 iii) 6 * (2 * 3)

   b) If p ⊕ q = $p^2 - q^2$, find: i) 5 ⊕ 3 ii) (4 ⊕ 2) ⊕ 1

2. **Properties:** a) For each operation, check if it is commutative and associative: i) a * b = a + b + ab ii) a * b = |a − b| iii) a * b = a^2b

   b) Prove algebraically that a * b = 2a + 2b − 3 is commutative.

3. **Identity and Inverse:** a) Find the identity element for: i) a * b = a + b − 5 ii) a * b = 3a + 3b iii) a * b = ab/2

   b) For a * b = a + b + 2 with identity −2, find the inverse of: i) 3 ii) −5 iii) 0

4. **Equation Solving:** Solve for x: a) x * 4 = 20, where a * b = 2a + b b) 3 * x = 7, where a * b = a^2 − b c) x * x = 15, where a * b = a + b + ab d) (x * 2) * 3 = 16, where a * b = ab − 1

5. **Operation Tables:** a) Construct the operation table for a * b = (a b) mod 5 on {0, 1, 2, 3, 4}. b) From your table, identify: i) Is it closed? ii) What is the identity element? iii) Which elements have inverses? iv) Is it commutative?

6. **Application Problem:** In a computer system, a binary operation ⊕ (XOR) is defined on {0, 1}:

   ```
   0 ⊕ 0 = 0
   0 ⊕ 1 = 1
   1 ⊕ 0 = 1
   1 ⊕ 1 = 0
   ```

   a) Construct the complete operation table b) Verify that it's commutative c) Verify that it's associative by testing all cases d) Find the identity element e) Find the inverse of each element f) Evaluate: (1 ⊕ 0) ⊕ (1 ⊕ 1)

WEEK 12: ARITHMETIC OF FINANCE

CONTENT

1. Simple Interest

Simple interest is interest calculated only on the principal amount.

Formula: I = (PRT)/100

or

I = PRT (when R is expressed as a decimal)

Where:

- **I** = Interest
- **P** = Principal (initial amount)
- **R** = Rate per annum (% per year)
- **T** = Time (in years)

Amount: A = P + I

or

A = P(1 + RT/100)

Example 1: Find the simple interest on ₦50,000 for 3 years at 8% per annum.

Solution:

```
P = ₦50,000
R = 8%
T = 3 years

I = PRT/100
I = (50,000 × 8 × 3)/100
I = 1,200,000/100
I = ₦12,000
```

Answer: ₦12,000

Example 2: A man borrows ₦80,000 at 5% simple interest per annum. How much will he repay after 4 years?

Solution:

```
P = ₦80,000
R = 5%
```

```
T = 4 years

I = (80,000 × 5 × 4)/100
I = 1,600,000/100
I = ₦16,000

Amount = P + I
      = 80,000 + 16,000
      = ₦96,000
```

Answer: ₦96,000

Example 3: At what rate will ₦2,500 amount to ₦3,000 in 4 years at simple interest?

Solution:

```
P = ₦2,500
A = ₦3,000
I = 3,000 – 2,500 = ₦500
T = 4 years

I = PRT/100
500 = (2,500 × R × 4)/100
500 = 10,000R/100
500 = 100R
R = 5%
```

Answer: 5% per annum

Example 4: How long will it take ₦5,000 to earn ₦1,500 as simple interest at 6% per annum?

Solution:

```
P = ₦5,000
I = ₦1,500
R = 6%

I = PRT/100
1,500 = (5,000 × 6 × T)/100
1,500 = 30,000T/100
1,500 = 300T
T = 5 years
```

Answer: 5 years

2. Compound Interest

Compound interest is interest calculated on the principal plus accumulated interest from previous periods.

Formula: $A = P(1 + r)^n$

Where:

- **A** = Final amount
- **P** = Principal
- **r** = Rate per period (as a decimal: R/100)
- **n** = Number of compounding periods

Compound Interest: $CI = A - P = P[(1 + r)^n - 1]$

Example 5: Find the compound interest on ₦10,000 for 3 years at 10% per annum compounded annually.

Solution:

```
P = ₦10,000
R = 10% = 0.10
n = 3 years

A = P(1 + r)ⁿ
A = 10,000(1 + 0.10)³
A = 10,000(1.10)³
A = 10,000(1.331)
A = ₦13,310

CI = A - P
CI = 13,310 - 10,000
CI = ₦3,310
```

Answer: ₦3,310

Example 6: A sum of ₦50,000 is invested at 8% per annum compound interest for 2 years. Find: a) The amount after 2 years b) The compound interest

Solution:

```
P = ₦50,000
r = 8% = 0.08
n = 2 years

a) A = 50,000(1.08)²
   A = 50,000(1.1664)
```

```
   A = ₦58,320

b) CI = 58,320 - 50,000
   CI = ₦8,320
```

Answer: a) ₦58,320 b) ₦8,320

Comparison: Simple vs. Compound Interest

Example 7: Compare simple and compound interest on ₦100,000 at 10% for 3 years.

Solution:

```
Simple Interest:
I = (100,000 × 10 × 3)/100 = ₦30,000
Amount = ₦130,000

Compound Interest:
A = 100,000(1.10)³ = 100,000(1.331) = ₦133,100
CI = 133,100 - 100,000 = ₦33,100

Difference = 33,100 - 30,000 = ₦3,100
```

Compound interest is ₦3,100 more.

3. Compound Interest with Different Compounding Periods

Interest can be compounded:

- **Annually:** Once per year (n = number of years)
- **Semi-annually:** Twice per year (n = 2 × number of years, r = annual rate ÷ 2)
- **Quarterly:** Four times per year (n = 4 × number of years, r = annual rate ÷ 4)
- **Monthly:** Twelve times per year (n = 12 × number of years, r = annual rate ÷ 12)

Modified Formula: A = P(1 + r/k)^(kt)

Where:

- **k** = number of times compounded per year
- **t** = time in years

Example 8: Find the amount when ₦20,000 is invested at 12% per annum for 2 years compounded: a) Annually b) Semi-annually c) Quarterly

Solution:

a) **Annually:**

$A = 20,000(1 + 0.12)^2$
$A = 20,000(1.12)^2$
$A = 20,000(1.2544)$
$A = ₦25,088$

b) **Semi-annually:**

Rate per period = 12% ÷ 2 = 6%
Number of periods = 2 × 2 = 4

$A = 20,000(1 + 0.06)^4$
$A = 20,000(1.06)^4$
$A = 20,000(1.2625)$
$A = ₦25,250$

c) **Quarterly:**

Rate per period = 12% ÷ 4 = 3%
Number of periods = 4 × 2 = 8

$A = 20,000(1 + 0.03)^8$
$A = 20,000(1.03)^8$
$A = 20,000(1.2668)$
$A = ₦25,336$

Note: More frequent compounding results in higher returns.

Example 9: A bank offers 10% per annum compounded monthly. How much will ₦100,000 grow to in 18 months?

Solution:

P = ₦100,000
Annual rate = 10%
Monthly rate = 10% ÷ 12 = 0.8333% = 0.008333
n = 18 months

$A = 100,000(1 + 0.008333)^{18}$
$A = 100,000(1.008333)^{18}$
$A = 100,000(1.1608)$
$A = ₦116,080$

Answer: ₦116,080

4. Annuities

An **annuity** is a series of equal payments made at regular intervals.

Types:

- **Ordinary Annuity:** Payments at end of each period
- **Annuity Due:** Payments at beginning of each period

Future Value of Ordinary Annuity: $FV = P \times [(1 + r)^n - 1]/r$

Where:

- **FV** = Future value
- **P** = Payment per period
- **r** = Interest rate per period
- **n** = Number of periods

Example 10: A person deposits ₦5,000 at the end of each year for 5 years in an account earning 8% per annum. Find the total amount at the end of 5 years.

Solution:

$P = ₦5,000$
$r = 8\% = 0.08$
$n = 5$ years

$FV = 5,000 \times [(1.08)^5 - 1]/0.08$
$FV = 5,000 \times [(1.4693 - 1)/0.08]$
$FV = 5,000 \times [0.4693/0.08]$
$FV = 5,000 \times 5.8666$
$FV = ₦29,333$

Answer: ₦29,333

Present Value of Annuity: $PV = P \times [1 - (1 + r)^{(-n)}]/r$

Example 11: What is the present value of receiving ₦10,000 annually for 4 years if the interest rate is 6%?

Solution:

$P = ₦10,000$
$r = 6\% = 0.06$
$n = 4$ years

$PV = 10,000 \times [1 - (1.06)^{(-4)}]/0.06$
$PV = 10,000 \times [1 - 0.7921]/0.06$

```
PV = 10,000 × [0.2079/0.06]
PV = 10,000 × 3.465
PV = ₦34,650
```

Answer: ₦34,650

5. Depreciation

Depreciation is the decrease in value of an asset over time.

Methods:

A. Straight-Line Method Equal depreciation each year.

Annual Depreciation = (Cost − Salvage Value)/Useful Life

Example 12: A machine costs ₦500,000 with a salvage value of ₦50,000 after 10 years. Find the annual depreciation.

Solution:

```
Annual Depreciation = (500,000 - 50,000)/10
                    = 450,000/10
                    = ₦45,000 per year
```

Answer: ₦45,000 per year

B. Reducing Balance Method (Declining Balance) Depreciation is a constant percentage of the book value.

Formula: $V = P(1 - r)^n$

Where:

- **V** = Value after n years
- **P** = Original cost
- **r** = Depreciation rate (as decimal)
- **n** = Number of years

Example 13: A car worth ₦2,000,000 depreciates at 15% per annum. Find its value after 3 years.

Solution:

```
P = ₦2,000,000
r = 15% = 0.15
n = 3 years

V = 2,000,000(1 - 0.15)³
```

```
V = 2,000,000(0.85)³
V = 2,000,000(0.6141)
V = ₦1,228,200
```

Answer: ₦1,228,200

Example 14: A laptop depreciates from ₦150,000 to ₦76,545 in 3 years. Find the annual rate of depreciation.

Solution:

```
P = ₦150,000
V = ₦76,545
n = 3 years

76,545 = 150,000(1 - r)³
(1 - r)³ = 76,545/150,000
(1 - r)³ = 0.5103
(1 - r) = ³√0.5103
(1 - r) = 0.7993
r = 1 - 0.7993
r = 0.2007
r = 20.07%
```

Answer: Approximately 20% per annum

6. Amortization

Amortization is the process of paying off a debt through regular payments over time. Each payment covers both interest and part of the principal.

Formula for Payment Amount: $PMT = P \times [r(1 + r)^n]/[(1 + r)^n - 1]$

Where:

- **PMT** = Payment per period
- **P** = Loan amount (principal)
- **r** = Interest rate per period
- **n** = Total number of payments

Example 15: A loan of ₦500,000 is to be repaid over 5 years at 10% per annum with monthly payments. Find the monthly payment.

Solution:

```
P = ₦500,000
Annual rate = 10%
Monthly rate r = 10%/12 = 0.008333
```

```
n = 5 × 12 = 60 months

PMT = 500,000 × [0.008333(1.008333)^60]/[(1.008333)^60 - 1]
PMT = 500,000 × [0.008333 × 1.6453]/[1.6453 - 1]
PMT = 500,000 × [0.0137]/[0.6453]
PMT = 500,000 × 0.02124
PMT = ₦10,620
```

Answer: ₦10,620 per month

Amortization Schedule: Shows how each payment is split between interest and principal.

Example 16: Create the first 3 months of an amortization schedule for a ₦100,000 loan at 12% annual interest (1% monthly) to be repaid in 12 months.

Solution:

First, find monthly payment:

```
P = 100,000
r = 0.01
n = 12

PMT = 100,000 × [0.01(1.01)^12]/[(1.01)^12 - 1]
PMT = 100,000 × [0.01 × 1.1268]/[0.1268]
PMT = 100,000 × 0.0889
PMT = ₦8,890
```

Amortization Schedule:

Month	Beginning Balance	Payment	Interest (1%)	Principal	Ending Balance
1	100,000	8,890	1,000	7,890	92,110
2	92,110	8,890	921	7,969	84,141
3	84,141	8,890	841	8,049	76,092

Calculations:

- Interest = Beginning Balance × rate
- Principal = Payment − Interest
- Ending Balance = Beginning Balance – Principal

7. Real-Life Financial Problems

Example 17: Savings Plan Amina wants to save ₦2,000,000 for her university education in 5 years. If a bank offers 9% per annum compounded annually, how much should she deposit now?

Solution:

```
A = ₦2,000,000
r = 9% = 0.09
n = 5 years

A = P(1 + r)ⁿ
2,000,000 = P(1.09)⁵
2,000,000 = P(1.5386)
P = 2,000,000/1.5386
P = ₦1,299,869
```

Answer: ₦1,299,869

Example 18: Investment Comparison Which is better:

- Option A: ₦100,000 at 12% simple interest for 3 years
- Option B: ₦100,000 at 10% compound interest for 3 years

Solution:

Option A (Simple Interest):

```
I = (100,000 × 12 × 3)/100 = ₦36,000
Amount = 100,000 + 36,000 = ₦136,000
```

Option B (Compound Interest):

```
A = 100,000(1.10)³
A = 100,000(1.331)
A = ₦133,100
```

Comparison: Option A yields ₦136,000 Option B yields ₦133,100

Answer: Option A is better by ₦2,900

Example 19: Mortgage Calculation A couple takes a mortgage of ₦10,000,000 at 8% annual interest for 20 years. Calculate: a) Monthly payment b) Total amount paid c) Total interest paid

Solution:

a) **Monthly payment:**

```
P = ₦10,000,000
r = 8%/12 = 0.006667
n = 20 × 12 = 240 months

PMT = 10,000,000 × [0.006667(1.006667)^240]/[(1.006667)^240 - 1]
PMT = 10,000,000 × [0.006667 × 4.9268]/[3.9268]
PMT = 10,000,000 × 0.008364
PMT = ₦83,640
```

b) **Total amount paid:**

```
Total = 83,640 × 240 = ₦20,073,600
```

c) **Total interest:**

```
Interest = 20,073,600 - 10,000,000 = ₦10,073,600
```

Answer: a) ₦83,640/month b) ₦20,073,600 c) ₦10,073,600

Example 20: Retirement Savings Bola deposits ₦20,000 at the end of each month for 25 years in a retirement account earning 7% per annum compounded monthly. How much will she have at retirement?

Solution:

```
P = ₦20,000
r = 7%/12 = 0.005833
n = 25 × 12 = 300 months

FV = 20,000 × [(1.005833)^300 - 1]/0.005833
FV = 20,000 × [(5.8086 - 1)/0.005833]
FV = 20,000 × [4.8086/0.005833]
FV = 20,000 × 824.56
FV = ₦16,491,200
```

Answer: ₦16,491,200

8. Inflation and Real Value

Real Interest Rate accounts for inflation: **Real rate ≈ Nominal rate − Inflation rate**

Example 21: If an investment yields 12% per annum but inflation is 8%, what is the real return?

Solution:

```
Real return = 12% - 8% = 4%
```

Answer: 4% (real terms)

Example 22: An item costs ₦50,000 today. If inflation is 7% per year, what will it cost in 3 years?

Solution:

```
Future cost = 50,000(1.07)³
            = 50,000(1.2250)
            = ₦61,250
```

Answer: ₦61,250

9. Investment Analysis

Return on Investment (ROI): ROI = (Gain − Cost)/Cost × 100%

Example 23: An investor buys shares for ₦200,000 and sells them for ₦250,000. Find the ROI.

Solution:

```
ROI = (250,000 - 200,000)/200,000 × 100%
    = 50,000/200,000 × 100%
    = 25%
```

Answer: 25%

Example 24: Break-even Analysis A business invests ₦5,000,000 in equipment. The equipment generates ₦800,000 profit annually. How long to break even if money could earn 10% elsewhere?

Solution:

This requires considering opportunity cost:

```
Annual profit needed = Investment × opportunity rate
                     = 5,000,000 × 0.10
                     = ₦500,000

Actual profit = ₦800,000
Net benefit = 800,000 - 500,000 = ₦300,000 per year

Time to recover investment:
5,000,000/300,000 = 16.67 years

Or simpler (without opportunity cost):
5,000,000/800,000 = 6.25 years
```

Answer: 6.25 years (simple) or 16.67 years (with opportunity cost)

10. Nigerian Financial Context

Example 25: Treasury Bills Nigerian Treasury Bills (T-Bills) are short-term government securities. A 91-day T-Bill with face value ₦1,000,000 is issued at a discount rate of 15% per annum. Calculate: a) Purchase price b) Interest earned

Solution:

a) **Purchase price:**

```
Discount for 91 days = (1,000,000 × 15 × 91)/(100 × 365)
                     = 1,365,000/36,500
                     = ₦37,397
```

```
Purchase price = 1,000,000 - 37,397
               = ₦962,603
```

b) **Interest earned:**

```
Interest = ₦37,397
```

Answer: a) ₦962,603 b) ₦37,397

Example 26: Fixed Deposit A bank offers 12% per annum on a fixed deposit with quarterly compounding. If ₦500,000 is deposited for 2 years: a) Find the maturity value b) Calculate effective annual rate

Solution:

a) **Maturity value:**

```
P = 500,000
r = 12%/4 = 3% per quarter
n = 2 × 4 = 8 quarters

A = 500,000(1.03)⁸
A = 500,000(1.2668)
A = ₦633,400
```

b) **Effective annual rate:**

```
Effective rate = (1 + 0.03)⁴ - 1
               = 1.1255 - 1
               = 0.1255 = 12.55%
```

Answer: a) ₦633,400 b) 12.55%

11. Practical Tips for Financial Planning

Rule of 72: To estimate doubling time: **Years to double ≈ 72/interest rate**

Example 27: At 9% interest, how long to double an investment?

Solution:

```
Time ≈ 72/9 = 8 years
```

Verification:

```
(1.09)⁸ = 1.9926 ≈ 2 ✓
```

Power of Compounding:

Example 28: Compare ₦10,000 invested at 10% for:

- 10 years
- 20 years
- 30 years

Solution:

```
10 years: 10,000(1.10)^10 = ₦25,937
20 years: 10,000(1.10)^20 = ₦67,275
30 years: 10,000(1.10)^30 = ₦174,494
```

Starting early makes a huge difference!

EVALUATION

1. Find the simple interest on ₦25,000 for 4 years at 6% per annum.
2. Calculate the compound interest on ₦40,000 for 3 years at 8% per annum.
3. Which gives more interest: ₦100,000 at 10% simple interest for 3 years, or ₦100,000 at 8% compound interest for 3 years?
4. A car worth ₦3,000,000 depreciates at 20% per annum. Find its value after 2 years.
5. Find the annual depreciation of a machine costing ₦800,000 with salvage value ₦80,000 after 10 years (straight-line method).
6. How much should be deposited now at 12% compound interest to have ₦500,000 in 4 years?
7. Find the monthly payment on a loan of ₦200,000 at 18% per annum for 2 years.
8. A person deposits ₦5,000 monthly for 3 years at 9% per annum. Find the total amount.
9. An investment of ₦150,000 grows to ₦180,000 in 2 years. Find the annual compound interest rate.
10. Using the Rule of 72, estimate how long it takes money to double at 8% interest.

ASSIGNMENT

1. **Simple and Compound Interest:** a) Calculate the simple interest on ₦75,000 for 5 years at 7.5% per annum.

   b) Find the compound interest on ₦60,000 for 2 years at 10% per annum compounded: i) Annually ii) Semi-annually iii) Quarterly

   c) A sum of money doubles itself in 8 years at simple interest. Find the rate of interest.

   d) At what compound interest rate will ₦50,000 amount to ₦66,550 in 3 years?

2. **Depreciation:** a) A laptop costs ₦200,000 and depreciates at 25% per annum. Find: i) Value after 3 years ii) Total depreciation after 3 years

   b) A machine depreciates from ₦500,000 to ₦320,000 in 2 years. Calculate: i) The annual rate of depreciation ii) Its value after 5 years

   c) Using straight-line depreciation, a vehicle costing ₦5,000,000 has salvage value ₦500,000 after 8 years. Find: i) Annual depreciation ii) Book value after 5 years

3. **Annuities and Savings:** a) Tunde saves ₦10,000 at the end of each month for 4 years at 12% per annum compounded monthly. Calculate the total amount saved.

   b) A woman wants to accumulate ₦5,000,000 in 10 years for her child's education. If she can invest at 11% per annum, how much should she deposit: i) As a lump sum now? ii) Monthly for 10 years?

   c) What is the present value of receiving ₦50,000 annually for 6 years at 8% discount rate?

4. **Loan and Mortgage:** a) A car loan of ₦2,500,000 is taken at 15% per annum for 5 years. i) Calculate the monthly payment ii) Find total amount repaid iii) Calculate total interest paid

   b) Create an amortization schedule for the first 4 months of a ₦500,000 loan at 18% per annum (1.5% monthly) for 2 years.

   c) A mortgage of ₦8,000,000 at 9% for 15 years requires what monthly payment?

5. **Investment Analysis:** a) Compare these two investment options for ₦1,000,000:

   - Option A: 13% simple interest for 4 years
   - Option B: 11% compound interest for 4 years (annually) Which is better and by how much?

   b) A business investment of ₦3,000,000 generates these annual profits: Year 1: ₦400,000 Year 2: ₦600,000 Year 3: ₦800,000 Year 4: ₦900,000

   Calculate: i) Total profit over 4 years ii) Average annual ROI iii) Is this better than investing at 12% compound interest?

6. **Real-World Applications:** a) A Treasury Bill with 182-day maturity and face value ₦5,000,000 is offered at 14% discount rate. Calculate: i) Purchase price ii) Interest earned iii) Effective yield

   b) Inflation is running at 10% per year. If your salary is ₦500,000 per year: i) What will you need to earn in 3 years to maintain the same purchasing power? ii) If you get 8% annual raises, what is your real income change?

   c) You invest ₦250,000 in a business. After 3 years, you sell your stake for ₦400,000. Calculate: i) Total return ii) Annual compound return rate iii) If you had invested at 15% compound interest instead, would you have earned more or less?

7. **Challenge Problem:** Chioma has ₦2,000,000 to invest. She has three options:

- **Option 1:** Fixed deposit at 12% per annum for 5 years (compound interest, annual)
- **Option 2:** Buy equipment for ₦2,000,000 that generates ₦600,000 annual profit but depreciates at 15% per year
- **Option 3:** Invest ₦500,000 in each of 4 different businesses:
  - Business A: 20% return per year for 5 years
  - Business B: Loses 10% first year, then 25% return for next 4 years
  - Business C: 15% return compounded quarterly
  - Business D: Simple interest at 18% per year

Calculate the final value for each option after 5 years and recommend the best investment strategy.

SS3 MATHEMATICS LESSON NOTES

SECOND TERM

WEEK 1: REVIEW OF FIRST TERM WORK & FINANCIAL MATHEMATICS

CONTENT

1. Introduction to Advanced Financial Concepts

Building on first term's work on simple interest, compound interest, and depreciation, we now explore more sophisticated financial instruments used in the real world.

2. Bonds and Debentures

A. Bonds

A **bond** is a debt security where the investor lends money to an entity (government or corporation) for a defined period at a fixed interest rate.

Key Terms:

- **Face Value (Par Value):** The amount the bond will be worth at maturity
- **Coupon Rate:** The annual interest rate paid on the bond
- **Maturity Date:** When the principal is repaid
- **Current Yield:** Annual interest ÷ Current market price

Example 1: A government bond has face value ₦100,000, coupon rate 12% per annum, and matures in 5 years. Calculate: a) Annual interest payment b) Total interest over the life of the bond

Solution:

```
a) Annual interest = Face value × Coupon rate
   = 100,000 × 12%
   = 100,000 × 0.12
   = ₦12,000 per year

b) Total interest = Annual interest × Number of years
   = 12,000 × 5
   = ₦60,000
```

Answer: a) ₦12,000 per year b) ₦60,000

Example 2: A bond with face value ₦50,000 and 10% coupon rate is selling at ₦45,000. Find the current yield.

Solution:

```
Annual interest = 50,000 × 10% = ₦5,000

Current yield = (Annual interest / Market price) × 100%
              = (5,000 / 45,000) × 100%
              = 11.11%
```

Answer: 11.11%

B. Debentures

A **debenture** is an unsecured debt instrument backed only by the creditworthiness of the issuer, not by physical assets.

Characteristics:

- No collateral required
- Fixed interest rate
- Priority over shareholders in case of liquidation
- Lower interest rates than secured bonds (generally)

Example 3: A company issues debentures worth ₦5,000,000 at 8% per annum for 10 years. Calculate: a) Annual interest payment b) Total repayment at maturity

Solution:

```
a) Annual interest = 5,000,000 × 8%
   = ₦400,000

b) Total repayment = Principal + Total interest
   = 5,000,000 + (400,000 × 10)
   = 5,000,000 + 4,000,000
   = ₦9,000,000
```

Answer: a) ₦400,000 b) ₦9,000,000

3. Shares and Dividends

Shares represent ownership in a company.

Types of Shares:

A. Ordinary Shares (Common Stock)

- Voting rights
- Dividends not guaranteed
- Last to be paid in liquidation
- Higher potential returns

B. Preference Shares (Preferred Stock)

- Fixed dividend rate
- Priority over ordinary shares
- Usually no voting rights
- Lower risk, lower returns

Key Terms:

- **Nominal Value (Par Value):** Face value of share
- **Market Value:** Current trading price
- **Dividend:** Portion of profits paid to shareholders
- **Dividend Yield:** (Dividend per share / Market price) × 100%

Example 4: An investor buys 500 shares at ₦40 per share. The company declares a dividend of ₦3 per share. Calculate: a) Total investment b) Total dividend received c) Dividend yield

Solution:

```
a) Total investment = 500 × 40 = ₦20,000

b) Total dividend = 500 × 3 = ₦1,500

c) Dividend yield = (3 / 40) × 100%
   = 7.5%
```

Answer: a) ₦20,000 b) ₦1,500 c) 7.5%

Example 5: A company with 1,000,000 shares of ₦10 nominal value declares 15% dividend. If you own 2,000 shares, how much dividend do you receive?

Solution:

```
Dividend per share = Nominal value × Dividend rate
                   = 10 × 15%
                   = ₦1.50

Total dividend = 2,000 × 1.50
               = ₦3,000
```

Answer: ₦3,000

Example 6: Shares with nominal value ₦50 are quoted at ₦65. The company pays 20% dividend. Find: a) Dividend per share b) Rate of return on investment

Solution:

```
a) Dividend per share = 50 × 20%
   = ₦10

b) Rate of return = (Dividend / Market price) × 100%
   = (10 / 65) × 100%
   = 15.38%
```

Answer: a) ₦10 b) 15.38%

4. Income Tax

Income tax is a percentage of income paid to the government.

Nigerian Tax System:

Personal Income Tax:

- Tax-free allowance (first portion not taxed)
- Progressive tax rates (higher income = higher rate)

Tax Relief: Various allowances reduce taxable income:

- Consolidated Relief Allowance
- Dependent allowances
- Life insurance relief
- National Housing Fund relief

Basic Formula:

```
Gross Income
- Allowable deductions
= Taxable Income
× Tax rate
= Tax payable
```

Example 7: A worker earns ₦3,600,000 per year. The first ₦300,000 is tax-free, and tax is charged at:

- 7% on next ₦300,000
- 11% on next ₦500,000
- 15% on next ₦500,000
- 19% on next ₦1,600,000
- 21% on remainder

Calculate the total tax payable.

Solution:

```
Taxable income = 3,600,000 - 300,000 = ₦3,300,000

Tax calculation:
On first 300,000:  300,000 × 7%  = ₦21,000
On next 500,000:   500,000 × 11% = ₦55,000
On next 500,000:   500,000 × 15% = ₦75,000
On next 1,600,000: 1,600,000 × 19% = ₦304,000
On remaining 400,000: 400,000 × 21% = ₦84,000

Total tax = 21,000 + 55,000 + 75,000 + 304,000 + 84,000
          = ₦539,000
```

Answer: ₦539,000

Example 8: An employee's annual gross salary is ₦2,400,000. After a consolidated relief of 20% + ₦200,000, tax is charged at:

- 7% on first ₦300,000
- 11% on next ₦300,000
- 15% on next ₦500,000
- 19% on remainder

Calculate: a) Taxable income b) Tax payable

Solution:

```
a) Gross salary = ₦2,400,000

   Relief = 20% of 2,400,000 + 200,000
          = 480,000 + 200,000
          = ₦680,000

   Taxable income = 2,400,000 - 680,000
                  = ₦1,720,000

b) Tax on first 300,000: 300,000 × 7% = ₦21,000
   Tax on next 300,000:  300,000 × 11% = ₦33,000
   Tax on next 500,000:  500,000 × 15% = ₦75,000
   Tax on remaining 620,000: 620,000 × 19% = ₦117,800

   Total tax = 21,000 + 33,000 + 75,000 + 117,800
             = ₦246,800
```

Answer: a) ₦1,720,000 b) ₦246,800

5. Value Added Tax (VAT)

VAT is a consumption tax added to the price of goods and services.

In Nigeria:

- Current VAT rate: 7.5% (as of 2020)
- Added to final selling price
- Collected by sellers, remitted to government

Formula:

```
VAT = Price × VAT rate
Final Price = Original Price + VAT
```

or

```
Final Price = Original Price × (1 + VAT rate)
```

Example 9: A laptop costs ₦250,000 before VAT. If VAT is 7.5%, find: a) VAT amount b) Total price

Solution:

```
a) VAT = 250,000 × 7.5%
       = 250,000 × 0.075
       = ₦18,750

b) Total price = 250,000 + 18,750
               = ₦268,750

Alternative: Total = 250,000 × 1.075 = ₦268,750
```

Answer: a) ₦18,750 b) ₦268,750

Example 10: A restaurant bill including 7.5% VAT is ₦21,500. Find the cost before VAT.

Solution:

```
Let original cost = x

x + 0.075x = 21,500
1.075x = 21,500
x = 21,500 / 1.075
x = ₦20,000

VAT = 21,500 - 20,000 = ₦1,500
```

Answer: ₦20,000 (VAT = ₦1,500)

Example 11: A business sells goods worth ₦5,000,000 in a month. How much VAT should be remitted to the government at 7.5%?

Solution:

```
VAT = 5,000,000 × 7.5%
    = ₦375,000
```

Answer: ₦375,000

6. Combined Financial Problems

Example 12: An investor purchases 1,000 shares at ₦45 each. The company pays 18% dividend on ₦50 nominal value. He also pays 10% tax on dividends. Calculate: a) Total investment b) Gross dividend c) Tax on dividend d) Net dividend received

Solution:

```
a) Total investment = 1,000 × 45 = ₦45,000

b) Dividend per share = 50 × 18% = ₦9
   Gross dividend = 1,000 × 9 = ₦9,000

c) Tax on dividend = 9,000 × 10% = ₦900

d) Net dividend = 9,000 - 900 = ₦8,100
```

Answer: a) ₦45,000 b) ₦9,000 c) ₦900 d) ₦8,100

Example 13: A trader buys goods for ₦500,000 and wants to make 25% profit. If VAT is 7.5%, find: a) Selling price before VAT b) VAT amount c) Final price to customer

Solution:

```
a) Cost = ₦500,000
   Profit = 500,000 × 25% = ₦125,000
   Selling price = 500,000 + 125,000 = ₦625,000

b) VAT = 625,000 × 7.5% = ₦46,875

c) Final price = 625,000 + 46,875 = ₦671,875
```

Answer: a) ₦625,000 b) ₦46,875 c) ₦671,875

7. Using Logarithm Tables for Financial Calculations

Logarithm tables can simplify complex financial calculations.

Example 14: Find the compound amount on ₦48,500 at 8% per annum for 5 years using logarithms.

Solution:

$A = P(1 + r)^n$
$A = 48{,}500(1.08)^5$

Taking log:
$\log A = \log 48{,}500 + 5 \log 1.08$

Using log tables:
$\log 48{,}500 = \log(4.85 \times 10^4) = \log 4.85 + 4$
$\qquad = 0.6857 + 4 = 4.6857$

$\log 1.08 = 0.0334$

Therefore:
$\log A = 4.6857 + 5(0.0334)$
$\qquad = 4.6857 + 0.1670$
$\qquad = 4.8527$

$A = \text{antilog}(4.8527)$
$\quad = 7.122 \times 10^4$
$\quad =$ ₦71,220

Answer: ₦71,220

Example 15: A sum amounts to ₦15,000 in 3 years at compound interest. Find the principal if the rate is 10% per annum (use logarithms).

Solution:

$A = P(1.10)^3$
$15{,}000 = P(1.10)^3$

$P = 15{,}000 / (1.10)^3$

Taking log:
$\log P = \log 15{,}000 - 3 \log 1.10$

$\log 15{,}000 = \log(1.5 \times 10^4) = 0.1761 + 4 = 4.1761$
$\log 1.10 = 0.0414$

$\log P = 4.1761 - 3(0.0414)$

```
      = 4.1761 - 0.1242
      = 4.0519

P = antilog(4.0519)
  = 1.127 × 10⁴
  = ₦11,270
```

Answer: ₦11,270

8. Practical Financial Planning

Example 16: A young professional earning ₦200,000 monthly wants to:

- Save 20% for investments
- Pay rent of ₦600,000 annually
- Set aside 10% for emergencies
- Budget the remainder

Calculate monthly allocations.

Solution:

```
Monthly income = ₦200,000
Annual income = 200,000 × 12 = ₦2,400,000

Savings (20%): 200,000 × 0.20 = ₦40,000/month

Rent (monthly equivalent): 600,000 ÷ 12 = ₦50,000/month

Emergency fund (10%): 200,000 × 0.10 = ₦20,000/month

Remainder = 200,000 - 40,000 - 50,000 - 20,000
          = ₦90,000/month for other expenses
```

Budget Summary:

- Savings: ₦40,000 (20%)
- Rent: ₦50,000 (25%)
- Emergency: ₦20,000 (10%)
- Living expenses: ₦90,000 (45%)

EVALUATION

1. A bond with face value ₦80,000 pays 9% annual interest. Find the annual interest payment.
2. An investor buys 800 shares at ₦35 each. If the company pays 12% dividend on ₦40 nominal value, calculate the total dividend received.

3. Calculate the VAT on goods worth ₦150,000 at 7.5% VAT rate.

4. A worker earns ₦1,800,000 annually. After ₦300,000 tax-free allowance, tax is 10% on the first ₦500,000 and 15% on the remainder. Find the tax payable.

5. A shop sells an item for ₦53,750 including 7.5% VAT. Find the price before VAT.

6. Shares with ₦25 nominal value pay 16% dividend. If purchased at ₦30, find the rate of return.

7. Use logarithms to find the amount when ₦25,000 is invested at 12% compound interest for 4 years.

8. A company issues debentures worth ₦2,000,000 at 7% for 8 years. Calculate the total interest payable.

9. Calculate the current yield on a ₦100,000 bond with 10% coupon rate selling at ₦95,000.

10. An investor owns 500 preference shares paying 8% dividend on ₦50 nominal value. Calculate the annual dividend.

ASSIGNMENT

1. **Bonds and Debentures:** a) A government bond with face value ₦250,000 and coupon rate 11% matures in 7 years. Calculate: i) Annual interest payment ii) Total interest over bond life iii) Total amount received at maturity

   b) A corporate bond with face value ₦100,000 and 9% coupon is selling at ₦92,000. Find: i) Current yield ii) If you buy 5 bonds, what annual interest do you receive?

2. **Shares and Dividends:** a) An investor buys 2,500 shares at ₦48 per share. The company declares 20% dividend on ₦40 nominal value. i) Find total investment ii) Calculate dividend per share iii) Find total dividend received iv) Calculate rate of return on investment

   b) A company has 5,000,000 shares of ₦10 nominal value. It declares 15% dividend. If you own 0.5% of the company, how much dividend do you receive?

3. **Income Tax:** a) Calculate tax on annual income of ₦4,200,000 with the following structure:

   - First ₦300,000: Tax-free
   - Next ₦300,000: 7%
   - Next ₦500,000: 11%
   - Next ₦500,000: 15%
   - Next ₦1,600,000: 19%
   - Remainder: 21%

   b) An employee earns ₦350,000 monthly. Annual consolidated relief is 20% + ₦200,000. Calculate: i) Annual gross income ii) Total relief iii) Taxable income iv) Tax payable using rates from (a)

4. **VAT Calculations:** a) Find VAT and total price for: i) Laptop: ₦180,000 (VAT 7.5%) ii) Restaurant meal: ₦12,500 (VAT 7.5%) iii) Car: ₦3,500,000 (VAT 7.5%)

   b) A shop's total sales including VAT are ₦5,375,000 for a month. Calculate: i) Sales before VAT ii) VAT amount to remit to government

5. **Combined Problems:** a) An investor's portfolio:

   - 1,000 shares (₦50 nominal, bought at ₦60, 18% dividend)
   - ₦500,000 in bonds (10% coupon rate)
   - All dividends and interest subject to 10% tax

   Calculate: i) Total investment in shares ii) Gross annual dividend from shares iii) Annual interest from bonds iv) Total gross income v) Tax on investment income vi) Net annual income

   b) A business scenario:

   - Buys goods for ₦800,000
   - Wants 30% gross profit
   - Must add 7.5% VAT
   - Pays 10% tax on profits

   Calculate: i) Selling price before VAT ii) VAT amount iii) Final customer price iv) Net profit after tax

6. **Using Logarithms:** a) Use logarithm tables to calculate: i) ₦75,500 at 9% compound interest for 6 years ii) Principal that amounts to ₦50,000 in 4 years at 12% compound interest

   b) A bond's value grows from ₦80,000 to ₦120,000 in 5 years. Find the annual compound growth rate using logarithms.

7. **Financial Planning:** Create a complete monthly budget for someone earning ₦300,000 per month:

   - 15% income tax
   - 25% for rent and utilities
   - 20% for savings and investments
   - 5% for insurance
   - 10% for transportation
   - Remainder for food and miscellaneous

   Show all calculations and create a pie chart showing the budget allocation.

WEEK 2: COORDINATE GEOMETRY OF A STRAIGHT LINE

CONTENT

1. The Cartesian Plane

The **Cartesian plane** (or coordinate plane) is a two-dimensional surface formed by two perpendicular number lines.

Components:

- **x-axis:** Horizontal number line
- **y-axis:** Vertical number line
- **Origin (O):** Point where axes intersect (0, 0)
- **Coordinates:** Ordered pair (x, y)

Diagram 1: The Cartesian Plane

```
      y-axis
        |
   Q2   |   Q1
        |
  ------+------   x-axis
        |
   Q3   |   Q4
        |
```

Quadrants:

- **Quadrant I (Q1):** $x > 0$, $y > 0$ (both positive)
- **Quadrant II (Q2):** $x < 0$, $y > 0$ (x negative, y positive)
- **Quadrant III (Q3):** $x < 0$, $y < 0$ (both negative)
- **Quadrant IV (Q4):** $x > 0$, $y < 0$ (x positive, y negative)

Example 1: Plot the following points and state their quadrants: A(3, 4), B(-2, 5), C(-3, -2), D(4, -3), E(0, 3), F(2, 0)

Solution:

```
          y
          |
    B     |     A
  •       |       •
          |
  ________|________   x
          |
      C   |        D
      •   |          •
          |
```

- A(3, 4): Quadrant I

- B(-2, 5): Quadrant II
- C(-3, -2): Quadrant III
- D(4, -3): Quadrant IV
- E(0, 3): On y-axis (not in any quadrant)
- F(2, 0): On x-axis (not in any quadrant)

2. Distance Between Two Points

The distance between points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is found using the **distance formula**:

Distance Formula: $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

Derivation: Using Pythagoras' theorem on the right triangle formed:

Diagram 2: Distance Between Points

```
    Q(x₂,y₂)
    •
    |\
    | \  d
    |  \
    •───•
    P   R
(x₁,y₁) (x₂,y₁)

Horizontal distance: |x₂ - x₁|
Vertical distance: |y₂ - y₁|
Hypotenuse: d
```

By Pythagoras: $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

Example 2: Find the distance between A(2, 3) and B(5, 7).

Solution:

```
Using d = √[(x₂ - x₁)² + (y₂ - y₁)²]

x₁ = 2, y₁ = 3
x₂ = 5, y₂ = 7

d = √[(5 - 2)² + (7 - 3)²]
  = √[3² + 4²]
  = √[9 + 16]
  = √25
  = 5 units
```

Answer: 5 units

Example 3: Find the distance between P(-3, 4) and Q(2, -8).

Solution:

$$
\begin{aligned}
d &= \sqrt{[(2 - (-3))^2 + (-8 - 4)^2]} \\
&= \sqrt{[(2 + 3)^2 + (-12)^2]} \\
&= \sqrt{[5^2 + 144]} \\
&= \sqrt{[25 + 144]} \\
&= \sqrt{169} \\
&= 13 \text{ units}
\end{aligned}
$$

Answer: 13 units

Example 4: The distance between points A(x, 3) and B(5, 7) is 5 units. Find the possible values of x.

Solution:

$$d = \sqrt{[(5 - x)^2 + (7 - 3)^2]} = 5$$

$$\sqrt{[(5 - x)^2 + 16]} = 5$$

Square both sides:

$$(5 - x)^2 + 16 = 25$$
$$(5 - x)^2 = 9$$
$$5 - x = \pm 3$$

If $5 - x = 3$:
$x = 2$

If $5 - x = -3$:
$x = 8$

Answer: x = 2 or x = 8

3. Midpoint of a Line Segment

The midpoint M of the line joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ is:

Midpoint Formula: $\mathbf{M = ((x_1 + x_2)/2, (y_1 + y_2)/2)}$

Diagram 3: Midpoint

P•————————•M————————•Q
(x_1,y_1) $\qquad\qquad\qquad$ (x_2,y_2)

M is exactly halfway between P and Q

Example 5: Find the midpoint of the line joining A(4, 6) and B(10, 14).

Solution:

```
M = ((4 + 10)/2, (6 + 14)/2)
  = (14/2, 20/2)
  = (7, 10)
```

Answer: (7, 10)

Example 6: The midpoint of the line joining P(2, y) and Q(8, 12) is M(5, 9). Find y.

Solution:

```
Midpoint formula: ((2 + 8)/2, (y + 12)/2) = (5, 9)

For x-coordinate:
(2 + 8)/2 = 5
10/2 = 5 ✓ (checks out)

For y-coordinate:
(y + 12)/2 = 9
y + 12 = 18
y = 6
```

Answer: y = 6

Example 7: A line segment has endpoints A(-3, 5) and B(7, -1). Find: a) The midpoint M b) The distance |AB| c) The distance |AM|

Solution:

```
a) M = ((-3 + 7)/2, (5 + (-1))/2)
     = (4/2, 4/2)
     = (2, 2)

b) |AB| = √[(7 - (-3))² + (-1 - 5)²]
        = √[10² + (-6)²]
        = √[100 + 36]
        = √136
        = 2√34 units

c) |AM| = √[(2 - (-3))² + (2 - 5)²]
        = √[5² + (-3)²]
        = √[25 + 9]
        = √34 units

Note: |AM| = |MB| = (1/2)|AB| ✓
```

Answer: a) (2, 2) b) 2√34 units c) √34 units

4. Gradient (Slope) of a Line

The **gradient** (or slope) measures the steepness of a line.

Gradient Formula: $m = (y_2 - y_1)/(x_2 - x_1)$ = rise/run

Diagram 4: Gradient

```
        Q(x₂,y₂)
       /•
      / |
     /  | rise = y₂ - y₁
    /   |
   •────┘
   P(x₁,y₁)
      run = x₂ - x₁
```

Types of Gradients:

- **Positive gradient ($m > 0$):** Line slopes upward (/) from left to right
- **Negative gradient ($m < 0$):** Line slopes downward () from left to right
- **Zero gradient ($m = 0$):** Horizontal line (—)
- **Undefined gradient:** Vertical line (|)

Example 8: Find the gradient of the line passing through A(2, 3) and B(6, 11).

Solution:

```
m = (11 - 3)/(6 - 2)
  = 8/4
  = 2
```

Answer: m = 2 (line slopes upward)

Example 9: Calculate the gradient of the line through P(-4, 7) and Q(2, -5).

Solution:

```
m = (-5 - 7)/(2 - (-4))
  = -12/(2 + 4)
  = -12/6
  = -2
```

Answer: m = −2 (line slopes downward)

Example 10: Find the gradient of the line joining points (3, k) and (7, 9) if the gradient is 2.

Solution:

$m = (9 - k)/(7 - 3) = 2$

$(9 - k)/4 = 2$
$9 - k = 8$
$k = 1$

Answer: $k = 1$

5. Special Cases of Gradients

A. Parallel Lines Lines are **parallel** if they have the **same gradient**.

If lines with gradients m_1 and m_2 are parallel: $\mathbf{m_1 = m_2}$

Example 11: Show that the line through A(1, 2) and B(4, 8) is parallel to the line through C(-2, 1) and D(1, 7).

Solution:

Gradient of AB:
$m_1 = (8 - 2)/(4 - 1) = 6/3 = 2$

Gradient of CD:
$m_2 = (7 - 1)/(1 - (-2)) = 6/3 = 2$

Since $m_1 = m_2 = 2$, the lines are parallel.

B. Perpendicular Lines Lines are **perpendicular** if the product of their gradients is **−1**.

If lines with gradients m_1 and m_2 are perpendicular: $\mathbf{m_1 \times m_2 = -1}$

or

$\mathbf{m_2 = -1/m_1}$ (negative reciprocal)

Example 12: Show that the line through P(2, 3) and Q(6, 5) is perpendicular to the line through R(1, 4) and S(3, 0).

Solution:

Gradient of PQ:
$m_1 = (5 - 3)/(6 - 2) = 2/4 = 1/2$

Gradient of RS:
$m_2 = (0 - 4)/(3 - 1) = -4/2 = -2$

Check: $m_1 \times m_2 = (1/2) \times (-2) = -1$ ✓

The lines are perpendicular.

Example 13: A line passes through A(3, 5) and B(7, k). Another line perpendicular to AB passes through C(2, 3) and D(6, 7). Find k.

Solution:

Gradient of CD:
$m_2 = (7 - 3)/(6 - 2) = 4/4 = 1$

Since AB ⊥ CD:
$m_1 \times m_2 = -1$
$m_1 \times 1 = -1$
$m_1 = -1$

For line AB:
$(k - 5)/(7 - 3) = -1$
$(k - 5)/4 = -1$
$k - 5 = -4$
$k = 1$

Answer: k = 1

6. Equation of a Straight Line

A. Slope-Intercept Form: y = mx + c

Where:

- **m** = gradient
- **c** = y-intercept (where line crosses y-axis)

Diagram 5: y = mx + c

```
     y
     |
     |  /
   c •-/———
     |/
  ———+———— x
     |
```

Example 14: Find the equation of a line with gradient 3 passing through (2, 5).

Solution:

```
Using y = mx + c:
m = 3, point (2, 5)

Substitute to find c:
5 = 3(2) + c
5 = 6 + c
c = -1

Equation: y = 3x - 1
```

Answer: y = 3x – 1

B. Two-Point Form: If line passes through (x_1, y_1) and (x_2, y_2):

$$(y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$$

Or simply find m, then use point-slope form.

Example 15: Find the equation of the line passing through A(1, 3) and B(4, 9).

Solution:

```
First find gradient:
m = (9 - 3)/(4 - 1) = 6/3 = 2

Using y = mx + c with point (1, 3):
3 = 2(1) + c
c = 1

Equation: y = 2x + 1
```

Verification with point (4, 9): y = 2(4) + 1 = 9 ✓

Answer: y = 2x + 1

C. Point-Slope Form: $y - y_1 = m(x - x_1)$

Where (x_1, y_1) is a known point and m is the gradient.

Example 16: Write the equation of a line with gradient −3 passing through (2, 5).

Solution:

```
Using y - y₁ = m(x - x₁):
y - 5 = -3(x - 2)
y - 5 = -3x + 6
y = -3x + 11
```

Answer: $y = -3x + 11$

D. General Form: $ax + by + c = 0$

Where a, b, c are constants.

7. Intercepts

x-intercept: Where line crosses x-axis ($y = 0$) **y-intercept:** Where line crosses y-axis ($x = 0$)

Example 17: Find the x and y intercepts of the line $3x + 4y = 12$.

Solution:

```
y-intercept (x = 0):
3(0) + 4y = 12
4y = 12
y = 3
y-intercept: (0, 3)

x-intercept (y = 0):
3x + 4(0) = 12
3x = 12
x = 4
x-intercept: (4, 0)
```

Answer: x-intercept (4, 0), y-intercept (0, 3)

Diagram 6: Graphing using intercepts

```
    y
  3 •
    |\
    | \
    |  \
────┼───•─── x
    0   4
```

Example 18: A line has equation $2x - 5y = 10$. Find: a) Gradient b) y-intercept c) x-intercept

Solution:

```
First rearrange to y = mx + c form:
-5y = -2x + 10
y = (2/5)x - 2

a) Gradient m = 2/5

b) y-intercept c = -2 (point (0, -2))

c) For x-intercept, put y = 0:
   2x - 5(0) = 10
   2x = 10
   x = 5
   x-intercept: (5, 0)
```

Answer: a) 2/5 b) (0, −2) c) (5, 0)

8. Angle of Inclination

The **angle of inclination** (θ) is the angle a line makes with the positive x-axis.

Relationship: m = tan θ

Diagram 7: Angle of Inclination

```
      /
     / θ
────•──────  x-axis
```

Example 19: Find the gradient of a line inclined at 45° to the x-axis.

Solution:

$$m = \tan 45° = 1$$

Answer: m = 1

Example 20: A line has gradient √3. Find its angle of inclination.

Solution:

$$m = \tan \theta = \sqrt{3}$$
$$\theta = \tan^{-1}(\sqrt{3})$$
$$\theta = 60°$$

Answer: 60°

9. Applications of Coordinate Geometry

Example 21: Collinearity Three points A(1, 2), B(3, 6), and C(5, 10) are collinear if they lie on the same straight line.

Check:

```
Gradient AB = (6 - 2)/(3 - 1) = 4/2 = 2
Gradient BC = (10 - 6)/(5 - 3) = 4/2 = 2

Since gradients are equal, points are collinear.
```

Example 22: Finding the fourth vertex Three vertices of a parallelogram are A(1, 2), B(4, 3), and C(6, 6). Find the fourth vertex D.

Solution: In a parallelogram, diagonals bisect each other.

Midpoint of AC = Midpoint of BD

```
Midpoint of AC = ((1 + 6)/2, (2 + 6)/2) = (3.5, 4)

Let D = (x, y)
Midpoint of BD = ((4 + x)/2, (3 + y)/2)

Setting equal:
(4 + x)/2 = 3.5  →  4 + x = 7  →  x = 3
(3 + y)/2 = 4    →  3 + y = 8  →  y = 5

D = (3, 5)
```

Answer: D(3, 5)

Example 23: Real-life application A road is being constructed between two towns. Town A is at coordinates (2, 5) km and Town B is at (10, 17) km on a map. Each unit represents 1 km.

Calculate: a) Distance between towns b) Gradient of the road c) If a rest stop is at the midpoint, find its coordinates

Solution:

```
a) Distance = √[(10 - 2)² + (17 - 5)²]
            = √[64 + 144]
            = √208
            = 4√13
            ≈ 14.42 km
```

```
b) Gradient = (17 – 5)/(10 – 2)
            = 12/8
            = 3/2 = 1.5

c) Midpoint = ((2 + 10)/2, (5 + 17)/2)
            = (6, 11)
```

Answer: a) 14.42 km b) 1.5 c) (6, 11)

EVALUATION

1. Plot the points A(3, 2), B(-2, 4), C(-3, -1), D(2, -3) and state their quadrants.
2. Find the distance between P(4, 7) and Q(10, 15).
3. Determine the midpoint of the line joining A(-4, 3) and B(6, -5).
4. Calculate the gradient of the line passing through (2, 5) and (8, 17).
5. Show that the points A(1, 2), B(3, 4), and C(7, 8) are collinear.
6. Find the equation of a line with gradient 4 passing through (1, 5).
7. Determine if the lines through A(1, 3), B(4, 5) and C(2, 1), D(8, -5) are parallel or perpendicular.
8. Find the x and y intercepts of $5x - 3y = 15$.
9. A line passes through (2, 7) and has gradient −3. Write its equation.
10. The points P(3, k), Q(7, 10) lie on a line with gradient 2. Find k.

ASSIGNMENT

1. **Distance and Midpoint:** a) Calculate the distance between: i) A(5, 8) and B(9, 11) ii) P(-3, 4) and Q(5, -2) iii) M(-6, -8) and N(2, 7)

   b) Find the midpoint of: i) The line joining (7, 9) and (15, 21) ii) AB where A(-5, 3) and B(7, -9)

   c) The distance between A(x, 5) and B(7, 9) is 5 units. Find possible values of x.

2. **Gradient:** a) Find the gradient of the line through: i) (3, 7) and (9, 19) ii) (-4, 5) and (2, -7) iii) (a, 3a) and (2a, 7a)

   b) Show that AB is parallel to CD if: A(2, 3), B(6, 7), C(-1, 2), D(3, 6)

   c) Prove that PQ is perpendicular to RS if: P(1, 4), Q(5, 6), R(3, 2), S(1, 6)

3. **Equations of Lines:** a) Find the equation of the line: i) With gradient 5 passing through (2, 3) ii) Through points (1, 4) and (3, 10) iii) With gradient −2 and y-intercept 5

b) Write in the form $y = mx + c$: i) $3x + 4y = 12$ ii) $2x - 5y + 10 = 0$ iii) $x - y = 7$

4. **Intercepts:** For each line, find x and y intercepts: a) $y = 3x - 6$ b) $2x + 3y = 12$ c) $5x - 4y = 20$ d) $y = -2x + 8$

5. **Applied Problems:** a) Three vertices of a rectangle are A(2, 1), B(6, 1), and C(6, 4). Find: i) The fourth vertex D ii) The length of diagonal AC iii) The midpoint of diagonal BD

   b) A treasure map shows treasure at T(8, 11). You start at S(2, 3). i) How far is the treasure? ii) If you walk to the midpoint first, what are its coordinates? iii) How much further to the treasure from the midpoint?

   c) Towns X, Y, Z are at coordinates (0, 0), (12, 5), (4, 9). i) Which two towns are closest? ii) If a hospital is built at the centroid (average of coordinates), find its location. iii) Find the equation of the road from X to Y.

6. **Proof Problems:** a) Prove that the triangle with vertices A(1, 2), B(4, 6), C(7, 2) is isosceles.

   b) Show that PQRS with P(1, 2), Q(5, 3), R(6, 7), S(2, 6) is a parallelogram by proving opposite sides are parallel.

   c) Prove that the points A(0, 0), B(4, 3), C(8, 6) are collinear.

WEEK 3: CONSTRUCTION I - LOCUS

CONTENT

1. Introduction to Locus

A **locus** (plural: loci) is the set of all points that satisfy a given condition or rule.

Definition: The locus of a point is the **path traced** by a point moving according to a given geometrical condition.

Examples:

- Locus of points equidistant from a fixed point is a **circle**
- Locus of points equidistant from two fixed points is the **perpendicular bisector** of the line joining them
- Locus of points equidistant from two intersecting lines is the **angle bisector**

2. Locus of Points Equidistant from a Fixed Point

Condition: All points at a constant distance from a fixed point O.

Result: A **circle** with center O and radius equal to the constant distance.

Diagram 8: Circle as Locus

```
        •
   •         •
 •      O     •
   •         •
        •

All points on circle are
equal distance (radius) from O
```

Construction Example 1: Construct the locus of points 4 cm from point P.

Steps:

1. Mark point P
2. Set compass to 4 cm
3. Place compass point on P
4. Draw a complete circle

Diagram 9:

```
     ___
   /     \
  |   P   |  ← 4 cm radius
   \ ___ /
```

```
All points on this circle
are exactly 4 cm from P
```

3. Locus of Points Equidistant from Two Fixed Points

Condition: All points equally distant from two fixed points A and B.

Result: The **perpendicular bisector** of line AB.

Diagram 10: Perpendicular Bisector

```
     |
     | ← perpendicular bisector
     |
A •──┼──• B
     M
```

```
Every point on the perpendicular
bisector is equidistant from A and B
```

Construction Example 2: Construct the locus of points equidistant from points A and B that are 6 cm apart.

Steps:

1. Draw line AB = 6 cm
2. Set compass to more than half of AB (e.g., 4 cm)
3. With center A, draw arcs above and below AB
4. With center B (same radius), draw arcs to intersect previous arcs
5. Join intersection points - this is the perpendicular bisector

Diagram 11: Construction Steps

```
      •P
     /|\
    / | \
   /  |  \
A •───┼───• B
   \  M  /
    \ | /
     \|/
      •Q
```

```
1. Arcs from A and B intersect at P and Q
2. Line PQ is perpendicular bisector
3. M is midpoint of AB
```

Proof that points on perpendicular bisector are equidistant: Take any point X on the perpendicular bisector.

- Triangle AXM ≅ Triangle BXM (SAS)
- Therefore: AX = BX ✓

4. Locus of Points Equidistant from Two Intersecting Lines

Condition: All points equally distant from two intersecting lines.

Result: The **angle bisectors** (two of them - one for each pair of opposite angles).

Diagram 12: Angle Bisector

```
    \  |  /
     \ | /
      \|/ ← angle bisector
  ─────┼─────
      /|\
     / | \
    /  |  \

Points on angle bisector are
equidistant from both lines
```

Construction Example 3: Construct the angle bisector of ∠ABC.

Steps:

1. Draw angle ABC
2. With center B, draw an arc cutting both arms at P and Q
3. With centers P and Q (same radius), draw arcs intersecting at R
4. Join BR - this is the angle bisector

Diagram 13: Angle Bisector Construction

```
      R
     /|\
    / | \
   /  |  \
  P   |   Q
   \  B  /
    \ | /
     \|/
  A───•───C

BR bisects ∠ABC
```

Property: Every point on the angle bisector is equidistant from the two arms of the angle.

5. Locus of Points Equidistant from a Fixed Line

Condition: All points at a constant distance from a line l.

Result: Two **parallel lines**, one on each side of l, at the given distance.

Diagram 14: Parallel Lines

```
────────────────    ← 2 cm from l

════════════════    ← line l (given)

────────────────    ← 2 cm from l

Both dashed lines are loci
```

Construction Example 4: Construct the locus of points 3 cm from a given line AB.

Steps:

1. Draw line AB
2. At several points on AB, construct perpendiculars
3. Mark points 3 cm from AB along each perpendicular
4. Join these points to form two parallel lines

6. Intersection of Loci

When two loci are constructed, their **intersection points** satisfy **both conditions**.

Example 24: Find the locus of points that are:

- 5 cm from point A
- 3 cm from point B Where A and B are 6 cm apart.

Solution:

1. Locus 1: Circle center A, radius 5 cm
2. Locus 2: Circle center B, radius 3 cm
3. Intersection: Two points where circles meet

Diagram 15: Intersecting Loci

```
    Circle (A, 5cm)
        ___
      /     \___
     |   A   |\  \ Circle (B, 3cm)
     |       | •X─•───
      \_____/ |  B    |
           ••Y        |
              \_____/
```

```
Points X and Y satisfy both conditions
```

Example 25: A goat is tied to a corner of a square shed (side 4m) with a 6m rope. Describe the locus of positions the goat can reach.

Solution: The locus consists of:

1. A three-quarter circle of radius 6m centered at the corner
2. Two quarter circles of radius 2m at the two adjacent corners

Diagram 16:

```
    Shed (4m × 4m)
     ┌─────┐
     │     │
     │     │ ←2m arc
 6m  │  •──┘
arc  └──────
```

```
The goat can graze in the shaded region
```

7. Circumscribed, Inscribed, and Escribed Circles

A. Circumscribed Circle (Circumcircle)

A circle that passes through all vertices of a polygon.

For a Triangle:

- Center: **Circumcenter** (intersection of perpendicular bisectors of sides)
- Passes through all three vertices

Diagram 17: Circumcircle

```
       B
      /•\
     / | \
    /  O  \ ← circumcircle
   /   |   \
  •────┼────•
  A    |    C
```

```
O is circumcenter
All vertices lie on circle
```

Construction Example 5: Construct the circumcircle of triangle ABC.

Steps:

1. Draw triangle ABC
2. Construct perpendicular bisector of AB
3. Construct perpendicular bisector of BC
4. Mark intersection point O (circumcenter)
5. With center O and radius OA, draw circle through A, B, C

Properties:

- Circumcenter is equidistant from all vertices
- For acute triangle: O is inside
- For right triangle: O is on hypotenuse midpoint
- For obtuse triangle: O is outside

B. Inscribed Circle (Incircle)

A circle that touches all sides of a polygon internally.

For a Triangle:

- Center: **Incenter** (intersection of angle bisectors)
- Tangent to all three sides

Diagram 18: Incircle

```
       B
      /.\
     /.|.\
    /  •  \ ← incircle
   / I/ \  \  touches all sides
  •─────────•
  A         C

I is incenter
Circle touches all sides
```

Construction Example 6: Construct the incircle of triangle ABC.

Steps:

1. Draw triangle ABC
2. Construct angle bisector of ∠A
3. Construct angle bisector of ∠B
4. Mark intersection point I (incenter)
5. From I, draw perpendicular to any side (say AB) meeting at P
6. With center I and radius IP, draw circle

Properties:

- Incenter is equidistant from all sides
- Always lies inside the triangle
- Angle bisectors always meet at one point

C. Escribed Circle (Excircle)

A circle that touches one side externally and the extensions of the other two sides.

For a Triangle:

- Three escribed circles possible (one for each side)
- Center: **Excenter** (intersection of one internal and two external angle bisectors)

Diagram 19: Escribed Circle

```
        B
       /|\
      / | \
     /  |  \
    /   |   \
   •─────────•
   A         C
    \   |   /
     \  •E /  ← escribed circle
      \_|_/    (opposite to B)
        |

Circle touches AC and
extensions of AB and BC
```

Construction is more complex and typically covered in advanced geometry.

8. Practical Construction Problems

Example 26: Two towns A and B are 8 cm apart on a map. A mobile phone tower is to be built such that it is:

- Equidistant from A and B
- 6 cm from A

Find possible locations for the tower using locus.

Solution:

```
Condition 1: Equidistant from A and B
→ Perpendicular bisector of AB

Condition 2: 6 cm from A
```

→ Circle center A, radius 6 cm

Solution: Intersection of these two loci

Construction:

1. Mark A and B, 8 cm apart
2. Construct perpendicular bisector of AB
3. Draw circle center A, radius 6 cm
4. Mark intersection points - these are possible locations

Diagram 20:

```
         •T₁
        /|
       / |
      /  |
A •──────┼──────• B
      \  |
       \ |
        \|
         •T₂
```

T_1 and T_2 are possible tower locations

Example 27: A garden is in the shape of a rectangle ABCD where AB = 10m, BC = 6m. A sprinkler is to be placed so that it waters points:

- Not more than 5m from corner A
- Not less than 4m from side BC

Shade the region where the sprinkler can be placed.

Solution:

Condition 1: Not more than 5m from A
→ Inside circle center A, radius 5m

Condition 2: Not less than 4m from BC
→ Above a line parallel to BC, 4m from BC

Solution: Region satisfying both conditions

Diagram 21:

```
B ────────────── C
│                │
│    shaded      │
│    region      │
│   (4m from BC) │
│ /─────\        │
│/   A   \       │6m
└────────────────  D

    10m

Shaded region shows where
sprinkler can be placed
```

EVALUATION

1. Define the term "locus" in geometry.
2. Describe the locus of points 5 cm from a fixed point O.
3. What is the locus of points equidistant from two fixed points A and B?
4. Construct the perpendicular bisector of a line segment PQ = 8 cm.
5. What is the locus of points equidistant from two intersecting lines?
6. Construct the angle bisector of an angle of 70°.
7. Distinguish between a circumcircle and an incircle of a triangle.
8. Where is the circumcenter of: a) An acute triangle? b) A right triangle? c) An obtuse triangle?
9. Two points A and B are 6 cm apart. Construct the locus of points that are 4 cm from A and 5 cm from B.
10. Describe the locus of points that are 2 cm from a line segment AB.

ASSIGNMENT

1. **Basic Locus Constructions:** a) Using ruler and compass, construct: i) The locus of points 4.5 cm from point P ii) The locus of points equidistant from points A and B that are 7 cm apart iii) The locus of points 3 cm from a line segment of length 8 cm

   b) Construct an angle of 60° and then construct its bisector. Verify that points on the bisector are equidistant from the two arms.

2. **Intersection of Loci:** a) Mark two points X and Y that are 8 cm apart. Construct the locus of points that are:

- 5 cm from X
- 6 cm from Y Mark the intersection points clearly.

b) Draw a line segment AB = 10 cm. Find and mark all points that are:

- 6 cm from A
- 3 cm from line AB

c) Two points P and Q are 9 cm apart. Find the locus of points that are:

- Equidistant from P and Q
- 5 cm from P Mark the two possible locations.

3. **Triangle Centers:** a) Draw a triangle ABC with AB = 8 cm, BC = 7 cm, AC = 6 cm. i) Construct the circumcircle and mark the circumcenter ii) Measure the circumradius iii) Verify that all vertices lie on the circle

   b) For the same triangle ABC: i) Construct the incircle and mark the incenter ii) Measure the inradius iii) Verify that the circle touches all three sides

4. **Applied Locus Problems:** a) A rectangular field ABCD has AB = 12m, BC = 8m. A scarecrow is to be placed such that it is:

   - At least 5m from corner A
   - Equidistant from sides AB and AD Using scale 1 cm = 2m, construct the locus and shade the possible region.

   b) Two radio stations A and B are 100 km apart. Station A has a range of 70 km and station B has a range of 60 km. i) Using scale 1 cm = 20 km, show the coverage area of each station ii) Shade the region covered by both stations iii) Find the locus of points equidistant from both stations iv) Mark points that are equidistant from both stations and exactly at the limit of both ranges

   c) A goat is tethered to the corner of a rectangular shed measuring 5m   3m with a 7m rope. Using scale 1 cm = 1m: i) Draw the shed ii) Construct the locus of points the goat can reach iii) Calculate the area the goat can graze (use π = 3.14)

5. **Challenge Problems:** a) Two towns A and B are 12 km apart. A police station is to be built such that:

   - It is closer to A than to B
   - It is not more than 8 km from A
   - It is at least 5 km from the road AB Using scale 1 cm = 2 km, construct and shade the region where the police station can be built.

   b) Triangle ABC has AB = 7 cm, BC = 8 cm, CA = 9 cm. i) Construct the triangle ii) Construct all three perpendicular bisectors and mark the circumcenter O iii) Construct all three angle bisectors and mark the incenter I iv) Measure the distance OI v) Draw both the circumcircle and incircle on the same diagram

6. **Proof and Reasoning:** a) Explain with a diagram why the perpendicular bisector of a line segment is the locus of points equidistant from the endpoints.

   b) Prove that the angle bisector is the locus of points equidistant from the two arms of the angle.

c) In triangle ABC, if the perpendicular bisectors of AB and BC meet at point O, prove that O is equidistant from all three vertices.

WEEK 4: DIFFERENTIATION OF ALGEBRAIC FUNCTIONS I

CONTENT

1. Introduction to Calculus

Calculus is the mathematics of change. It has two main branches:

- **Differential Calculus:** Rates of change (differentiation)
- **Integral Calculus:** Accumulation (integration)

Differentiation answers the question: "How fast is something changing?"

Real-life examples:

- Speed: rate of change of distance with respect to time
- Acceleration: rate of change of velocity with respect to time
- Population growth rate
- Rate of cooling of hot water
- Marginal cost in economics

2. Concept of a Limit

A **limit** describes the value a function approaches as the input approaches some value.

Notation: lim[x→a] f(x) = L

Read as: "The limit of f(x) as x approaches a is L"

Example 28: Evaluate lim[x→2] ($x^2 + 3x$)

Solution:

```
Simply substitute x = 2:
= (2)² + 3(2)
= 4 + 6
= 10
```

Answer: 10

Example 29: Evaluate lim[x→3] ($x^2 - 9$)/($x - 3$)

Solution:

```
Direct substitution gives 0/0 (indeterminate)

Factor numerator:
(x² - 9)/(x - 3) = (x + 3)(x - 3)/(x - 3)
                 = x + 3  (for x ≠ 3)
```

Now take limit:

$$\lim_{x \to 3} (x + 3) = 3 + 3 = 6$$

Answer: 6

Example 30: Evaluate $\lim_{h \to 0} [(2 + h)^2 - 4]/h$

Solution:

Expand numerator:

$$(2 + h)^2 - 4 = 4 + 4h + h^2 - 4 = 4h + h^2$$

Therefore:

$$[4h + h^2]/h = [h(4 + h)]/h = 4 + h \quad (\text{for } h \neq 0)$$

$$\lim_{h \to 0} (4 + h) = 4$$

Answer: 4

This type of limit is the foundation of differentiation!

3. The Concept of a Derivative

The **derivative** of a function at a point gives the **rate of change** or **slope of the tangent** at that point.

Geometric Interpretation:

Diagram 22: Slope of Secant vs. Tangent

```
      y
      |      • Q(x+h, f(x+h))
      |     /|
      |    / |
      |   /  |
      |  /   |
      |/____| P(x, f(x))
      •
    __|_________x
```

As Q moves closer to P ($h \to 0$),
secant PQ approaches tangent at P

Slope of secant PQ:

$$m_{secant} = [f(x + h) - f(x)]/h$$

Slope of tangent at P (derivative):

$$m_{tangent} = \lim_{h \to 0} [f(x + h) - f(x)]/h$$

This is the **first derivative** of f(x), denoted f'(x) or dy/dx.

4. First Principle of Differentiation (Definition)

First Principle (Derivative from First Principles):

$f'(x) = \lim_{h\to 0} [f(x + h) - f(x)]/h$

Alternative Notation:

- **f'(x):** "f prime of x"
- **dy/dx:** "dy by dx" or "derivative of y with respect to x"
- **d/dx[f(x)]:** "derivative of f(x) with respect to x"

Example 31: Find the derivative of $f(x) = x^2$ from first principles.

Solution:

$$f'(x) = \lim_{h\to 0} [f(x + h) - f(x)]/h$$

$$\begin{aligned} f(x) &= x^2 \\ f(x + h) &= (x + h)^2 \\ &= x^2 + 2xh + h^2 \end{aligned}$$

Therefore:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} [(x^2 + 2xh + h^2) - x^2]/h \\ &= \lim_{h\to 0} [2xh + h^2]/h \\ &= \lim_{h\to 0} [h(2x + h)]/h \\ &= \lim_{h\to 0} (2x + h) \\ &= 2x \end{aligned}$$

Answer: $f'(x) = 2x$

Interpretation: If $f(x) = x^2$, then $f'(x) = 2x$

- At $x = 3$, slope $= 2(3) = 6$
- At $x = 5$, slope $= 2(5) = 10$

Example 32: Differentiate $f(x) = x^3$ from first principles.

Solution:

$$\begin{aligned} f(x + h) &= (x + h)^3 \\ &= x^3 + 3x^2h + 3xh^2 + h^3 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} [(x^3 + 3x^2h + 3xh^2 + h^3) - x^3]/h \\ &= \lim_{h\to 0} [3x^2h + 3xh^2 + h^3]/h \\ &= \lim_{h\to 0} [h(3x^2 + 3xh + h^2)]/h \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\ &= 3x^2 \end{aligned}$$

Answer: $f'(x) = 3x^2$

Example 33: Find f'(x) if f(x) = 2x + 5 using first principles.

Solution:

```
f(x + h) = 2(x + h) + 5
         = 2x + 2h + 5

f'(x) = lim[h→0] [(2x + 2h + 5) − (2x + 5)]/h
      = lim[h→0] [2h]/h
      = lim[h→0] 2
      = 2
```

Answer: f'(x) = 2 (constant slope - straight line)

Example 34: Differentiate f(x) = 1/x from first principles.

Solution:

```
f(x + h) = 1/(x + h)

f'(x) = lim[h→0] [1/(x + h) − 1/x]/h
      = lim[h→0] [x − (x + h)]/[h·x(x + h)]
      = lim[h→0] [−h]/[h·x(x + h)]
      = lim[h→0] [−1]/[x(x + h)]
      = −1/x²
```

Answer: $f'(x) = -1/x^2$

5. Standard Derivatives (Derived Functions)

From first principles, we can derive standard formulas:

Power Rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

Table of Standard Derivatives:

Function f(x)	Derivative f'(x)
c (constant)	0
x	1
x^2	2x
x^3	$3x^2$
x^n	nx^{n-1}
$1/x = x^{-1}$	$-1/x^2 = -x^{-2}$
$\sqrt{x} = x^{(1/2)}$	$1/(2\sqrt{x}) = (1/2)x^{-(1/2)}$
$1/x^2 = x^{-2}$	$-2/x^3 = -2x^{-3}$

Rules of Differentiation:

1. Constant Multiple Rule: d/dx[cf(x)] = c·f'(x)

2. Sum Rule: d/dx[f(x) + g(x)] = f'(x) + g'(x)

3. Difference Rule: d/dx[f(x) − g(x)] = f'(x) − g'(x)

Example 35: Differentiate $y = 5x^4$

Solution:

```
Using power rule with constant multiple:
dy/dx = 5 × 4x³
      = 20x³
```

Answer: $dy/dx = 20x^3$

Example 36: Find $f'(x)$ if $f(x) = 3x^2 + 4x - 7$

Solution:

```
Differentiate term by term:
d/dx[3x²] = 3 × 2x = 6x
d/dx[4x] = 4 × 1 = 4
d/dx[-7] = 0

Therefore:
f'(x) = 6x + 4
```

Answer: $f'(x) = 6x + 4$

Example 37: Differentiate $y = 2x^5 - 3x^3 + 7x^2 - 5x + 9$

Solution:

```
dy/dx = 2(5x⁴) - 3(3x²) + 7(2x) - 5(1) + 0
      = 10x⁴ - 9x² + 14x - 5
```

Answer: $dy/dx = 10x^4 - 9x^2 + 14x - 5$

Example 38: Find dy/dx if $y = (x + 2)(x - 3)$

Solution:

Method 1: Expand first

```
y = x² - 3x + 2x - 6
  = x² - x - 6

dy/dx = 2x - 1
```

Method 2: Product rule (covered later)

Answer: dy/dx = 2x − 1

Example 39: Differentiate $y = (3x^2 + 5)/x$

Solution:

```
First simplify by dividing:
y = 3x²/x + 5/x
  = 3x + 5x⁻¹

dy/dx = 3(1) + 5(-1)x⁻²
      = 3 - 5/x²
```

Answer: dy/dx = $3 - 5/x^2$

Example 40: Find f'(x) if $f(x) = 2\sqrt{x} + 3/\sqrt{x}$

Solution:

```
Rewrite using powers:
f(x) = 2x^(1/2) + 3x^(-1/2)

f'(x) = 2(1/2)x^(-1/2) + 3(-1/2)x^(-3/2)
      = x^(-1/2) - (3/2)x^(-3/2)
      = 1/√x - 3/(2x√x)
      = 1/√x - 3/(2x^(3/2))
```

Answer: $f'(x) = 1/\sqrt{x} - 3/(2x\sqrt{x})$

6. Interpretation of Derivatives

Physical Interpretation:

- **Distance-Time:** If s(t) = distance, then s'(t) = velocity
- **Velocity-Time:** If v(t) = velocity, then v'(t) = acceleration

Geometric Interpretation:

- f'(a) = slope of tangent to curve y = f(x) at x = a
- f'(a) = instantaneous rate of change at x = a

Example 41: A particle moves along a line according to $s(t) = t^3 - 6t^2 + 9t$, where s is in meters and t is in seconds. Find: a) Velocity at any time t b) Velocity at t = 2 seconds c) When is the particle at rest?

Solution:

```
a) Velocity v(t) = ds/dt
   v(t) = 3t² - 12t + 9

b) At t = 2:
   v(2) = 3(2)² - 12(2) + 9
        = 12 - 24 + 9
        = -3 m/s

   (Negative means moving backward)

c) Particle at rest when v = 0:
   3t² - 12t + 9 = 0
   t² - 4t + 3 = 0
   (t - 1)(t - 3) = 0
   t = 1 or t = 3 seconds
```

Answer: a) $v(t) = 3t^2 - 12t + 9$ b) −3 m/s c) t = 1s and t = 3s

Example 42: The displacement of a body is given by $s = 2t^2 + 3t - 5$ where s is in cm and t in seconds. Find: a) Initial velocity b) Velocity after 4 seconds c) Acceleration (rate of change of velocity)

Solution:

```
a) Velocity v = ds/dt = 4t + 3
   At t = 0: v(0) = 4(0) + 3 = 3 cm/s

b) At t = 4: v(4) = 4(4) + 3 = 19 cm/s

c) Acceleration a = dv/dt = d/dt[4t + 3] = 4 cm/s²
```

Answer: a) 3 cm/s b) 19 cm/s c) 4 cm/s²

7. Finding Equations of Tangents and Normals

Tangent: Line that touches a curve at exactly one point (locally).

Normal: Line perpendicular to the tangent at the point of contact.

Steps to find tangent equation at point (x_1, y_1):

1. Find dy/dx (general derivative)
2. Substitute $x = x_1$ to find slope m at that point
3. Use point-slope form: $y - y_1 = m(x - x_1)$

For normal:

- Slope of normal = −1/(slope of tangent) = −1/m
- Use same point-slope form with new slope

Example 43: Find the equation of the tangent to $y = x^2$ at the point (3, 9).

Solution:

```
dy/dx = 2x

At x = 3:
m = 2(3) = 6

Equation of tangent:
y - 9 = 6(x - 3)
y - 9 = 6x - 18
y = 6x - 9
```

Answer: y = 6x – 9

Example 44: Find equations of both tangent and normal to $y = x^3 - 3x$ at $x = 2$.

Solution:

```
First find y-coordinate:
y = (2)³ - 3(2) = 8 - 6 = 2
Point is (2, 2)

Find slope:
dy/dx = 3x² - 3
At x = 2: m = 3(4) - 3 = 9

**Tangent:**
y - 2 = 9(x - 2)
y = 9x - 16

Normal:
Slope of normal = -1/9

y - 2 = (-1/9)(x - 2)
9y - 18 = -x + 2
9y = -x + 20
x + 9y = 20
```

Answer: Tangent: $y = 9x - 16$; Normal: $x + 9y = 20$

EVALUATION

1. Evaluate: $\lim[x\to5]\ (x^2 + 2x)$
2. Evaluate: $\lim[h\to0]\ [(x + h)^2 - x^2]/h$
3. Differentiate $y = x^4$ from first principles.
4. Find $f'(x)$ if $f(x) = 3x^2 + 5x - 2$.
5. Differentiate: $y = 4x^3 - 7x^2 + 2x - 9$
6. If $f(x) = x^5$, find $f'(2)$.
7. Find dy/dx if $y = (2x^2 + 3)/x$.
8. A particle's position is $s(t) = t^2 + 4t$. Find its velocity at $t = 3$.
9. Find the slope of the tangent to $y = x^3$ at $x = 1$.
10. Differentiate $y = 3\sqrt{x} + 2/x$.

ASSIGNMENT

1. **Limits:** Evaluate: a) $\lim[x\to4]\ (x^2 - 16)/(x - 4)$ b) $\lim[h\to0]\ [(3 + h)^2 - 9]/h$ c) $\lim[x\to2]\ (x^3 - 8)/(x - 2)$ d) $\lim[h\to0]\ [(1 + h)^3 - 1]/h$
2. **First Principles:** Differentiate from first principles: a) $f(x) = 3x + 2$ b) $f(x) = x^2 + 4x$ c) $f(x) = 2x^3$ d) $f(x) = 1/(x + 1)$
3. **Standard Differentiation:** Find dy/dx for: a) $y = 5x^4 - 3x^2 + 7$ b) $y = 2x^3 + 4x^2 - 6x + 1$ c) $y = x^5 - 2x^4 + 3x^3 - x$ d) $y = (x + 1)(x^2 - 3)$ e) $y = (3x^3 - 4x)/x^2$ f) $y = 4\sqrt{x} - 3/\sqrt{x} + 2x$
4. **Applications - Motion:** a) A particle moves according to $s = t^3 - 9t^2 + 15t$ where s is in meters and t in seconds. i) Find velocity as a function of time ii) Find velocity at $t = 2$ iii) When is the particle at rest? iv) Find acceleration

   b) The height of a ball thrown upward is $h = 20t - 5t^2$ meters after t seconds. i) Find the initial velocity ii) When does the ball reach maximum height? iii) What is the maximum height?
5. **Tangents and Normals:** a) Find the equation of the tangent to $y = x^2 + 2x$ at $(1, 3)$.

   b) Find equations of tangent and normal to $y = x^3 - 3x^2$ at $x = 2$.

   c) At what point on the curve $y = x^2$ is the tangent parallel to the line $y = 4x - 1$?

   d) Find the points on $y = x^3$ where the tangent is horizontal.

6. **Challenge Problems:** a) If $f(x) = ax^2 + bx + c$ and $f'(1) = 8$, $f'(2) = 14$, $f(0) = 5$, find a, b, and c.

b) The curve $y = x^3 + px + q$ passes through (1, 6) and has slope 5 at that point. Find p and q.

c) Prove from first principles that if $f(x) = x^n$, then $f'(x) = nx^{n-1}$ for $n = 4$.

WEEK 6: DIFFERENTIATION OF ALGEBRAIC FUNCTIONS II

CONTENT

1. Review of Basic Differentiation

From Week 4, recall:

- **Power Rule:** $d/dx(x^n) = nx^{n-1}$
- **Constant Multiple:** $d/dx(cf(x)) = c \cdot f'(x)$
- **Sum/Difference:** $d/dx(f \pm g) = f' \pm g'$

Now we extend to more advanced techniques and applications.

2. Product Rule

When differentiating a product of two functions, we cannot simply multiply their derivatives. We need the **Product Rule**.

Product Rule: If $y = u \cdot v$ where u and v are functions of x, then: $dy/dx = u(dv/dx) + v(du/dx)$

Or in short notation: **$d/dx(uv) = u \cdot v' + v \cdot u'$**

Memory aid: "First times derivative of second, plus second times derivative of first"

Example 1: Differentiate $y = (3x^2 + 2)(4x^3 - 5)$

Solution:

Method 1: Using Product Rule

```
Let u = 3x² + 2  →  du/dx = 6x
Let v = 4x³ - 5  →  dv/dx = 12x²

dy/dx = u(dv/dx) + v(du/dx)
      = (3x² + 2)(12x²) + (4x³ - 5)(6x)
      = 36x⁴ + 24x² + 24x⁴ - 30x
      = 60x⁴ + 24x² - 30x
```

Method 2: Expand first, then differentiate

```
y = 12x⁵ - 15x² + 8x³ - 10
dy/dx = 60x⁴ + 24x² - 30x
```

Both methods give the same answer ✓

Answer: $dy/dx = 60x^4 + 24x^2 - 30x$

Example 2: Find dy/dx if $y = x^2(x + 1)^4$

Solution:

Here we must use the product rule (expanding would be tedious).

$u = x^2 \rightarrow u' = 2x$
$v = (x + 1)^4 \rightarrow v' = 4(x + 1)^3$ (chain rule - covered next)

$$\begin{aligned} dy/dx &= x^2 \cdot 4(x + 1)^3 + (x + 1)^4 \cdot 2x \\ &= 4x^2(x + 1)^3 + 2x(x + 1)^4 \\ &= 2x(x + 1)^3[2x + (x + 1)] \\ &= 2x(x + 1)^3(3x + 1) \end{aligned}$$

Answer: $dy/dx = 2x(x + 1)^3(3x + 1)$

Example 3: Differentiate $y = (2x + 3)(x^2 - 5x + 1)$

Solution:

$u = 2x + 3 \rightarrow u' = 2$
$v = x^2 - 5x + 1 \rightarrow v' = 2x - 5$

$$\begin{aligned} dy/dx &= (2x + 3)(2x - 5) + (x^2 - 5x + 1)(2) \\ &= 4x^2 - 10x + 6x - 15 + 2x^2 - 10x + 2 \\ &= 6x^2 - 14x - 13 \end{aligned}$$

Answer: $dy/dx = 6x^2 - 14x - 13$

3. Quotient Rule

When differentiating a quotient (fraction), we use the **Quotient Rule**.

Quotient Rule: If $y = u/v$, then: $dy/dx = [v(du/dx) - u(dv/dx)]/v^2$

Or in short: **$d/dx(u/v) = (v \cdot u' - u \cdot v')/v^2$**

Memory aid: "Bottom times derivative of top, minus top times derivative of bottom, all over bottom squared"

Example 4: Differentiate $y = (3x + 2)/(x - 1)$

Solution:

$u = 3x + 2 \rightarrow u' = 3$
$v = x - 1 \rightarrow v' = 1$

$$\begin{aligned} dy/dx &= [(x - 1)(3) - (3x + 2)(1)]/(x - 1)^2 \\ &= [3x - 3 - 3x - 2]/(x - 1)^2 \\ &= -5/(x - 1)^2 \end{aligned}$$

Answer: $dy/dx = -5/(x - 1)^2$

Example 5: Find $f'(x)$ if $f(x) = x^2/(x + 3)$

Solution:

$$u = x^2 \rightarrow u' = 2x$$
$$v = x + 3 \rightarrow v' = 1$$

$$\begin{aligned} f'(x) &= [(x + 3)(2x) - x^2(1)]/(x + 3)^2 \\ &= [2x^2 + 6x - x^2]/(x + 3)^2 \\ &= [x^2 + 6x]/(x + 3)^2 \\ &= x(x + 6)/(x + 3)^2 \end{aligned}$$

Answer: $f'(x) = x(x + 6)/(x + 3)^2$

Example 6: Differentiate $y = (x^2 + 1)/(x^2 - 1)$

Solution:

$$u = x^2 + 1 \rightarrow u' = 2x$$
$$v = x^2 - 1 \rightarrow v' = 2x$$

$$\begin{aligned} dy/dx &= [(x^2 - 1)(2x) - (x^2 + 1)(2x)]/(x^2 - 1)^2 \\ &= [2x^3 - 2x - 2x^3 - 2x]/(x^2 - 1)^2 \\ &= -4x/(x^2 - 1)^2 \end{aligned}$$

Answer: $dy/dx = -4x/(x^2 - 1)^2$

4. Chain Rule (Function of a Function)

When a function is composed of one function inside another, we use the **Chain Rule**.

Chain Rule: If $y = f(u)$ where $u = g(x)$, then: $dy/dx = (dy/du) \times (du/dx)$

Or: **$d/dx[f(g(x))] = f'(g(x))\cdot g'(x)$**

In words: Differentiate the outer function (keeping the inner function unchanged), then multiply by the derivative of the inner function.

Example 7: Differentiate $y = (3x + 2)^5$

Solution:

$$\text{Let } u = 3x + 2 \rightarrow du/dx = 3$$
$$\text{Then } y = u^5 \rightarrow dy/du = 5u^4$$

$$\begin{aligned} dy/dx &= (dy/du) \times (du/dx) \\ &= 5u^4 \times 3 \\ &= 15u^4 \\ &= 15(3x + 2)^4 \end{aligned}$$

Answer: $dy/dx = 15(3x + 2)^4$

Example 8: Find dy/dx if $y = (x^2 - 3x + 1)^8$

Solution:

```
Outer function: ( )⁸  →  derivative: 8( )⁷
Inner function: x² − 3x + 1  →  derivative: 2x − 3

dy/dx = 8(x² − 3x + 1)⁷ × (2x − 3)
      = 8(2x − 3)(x² − 3x + 1)⁷
```

Answer: $dy/dx = 8(2x - 3)(x^2 - 3x + 1)^7$

Example 9: Differentiate $y = \sqrt{4x^2 + 1}$

Solution:

```
Rewrite: y = (4x² + 1)^(1/2)

Outer function: ( )^(1/2)  →  (1/2)( )^(-1/2)
Inner function: 4x² + 1  →  8x

dy/dx = (1/2)(4x² + 1)^(-1/2) × 8x
      = 4x/(4x² + 1)^(1/2)
      = 4x/√(4x² + 1)
```

Answer: $dy/dx = 4x/\sqrt{4x^2 + 1}$

Example 10: Find $f'(x)$ if $f(x) = 1/(2x - 3)^2$

Solution:

```
Rewrite: f(x) = (2x - 3)^(-2)

Outer: ( )^(-2)  →  -2( )^(-3)
Inner: 2x - 3  →  2

f'(x) = -2(2x - 3)^(-3) × 2
      = -4(2x - 3)^(-3)
      = -4/(2x - 3)³
```

Answer: $f'(x) = -4/(2x - 3)^3$

5. Combined Rules

Often we need to combine product, quotient, and chain rules.

Example 11: Differentiate $y = x^2(3x - 1)^4$

Solution:

```
Use Product Rule with Chain Rule:

u = x²  →  u' = 2x
```

$v = (3x - 1)^4 \rightarrow v' = 4(3x - 1)^3 \times 3 = 12(3x - 1)^3$

$$\begin{aligned}
dy/dx &= x^2 \times 12(3x - 1)^3 + (3x - 1)^4 \times 2x \\
&= 12x^2(3x - 1)^3 + 2x(3x - 1)^4 \\
&= 2x(3x - 1)^3[6x + (3x - 1)] \\
&= 2x(3x - 1)^3(9x - 1)
\end{aligned}$$

Answer: $dy/dx = 2x(3x - 1)^3(9x - 1)$

Example 12: Find dy/dx if $y = (x + 1)^3/(x - 1)^2$

Solution:

Use Quotient Rule with Chain Rule:

$u = (x + 1)^3 \rightarrow u' = 3(x + 1)^2$
$v = (x - 1)^2 \rightarrow v' = 2(x - 1)$

$$\begin{aligned}
dy/dx &= [(x - 1)^2 \times 3(x + 1)^2 - (x + 1)^3 \times 2(x - 1)]/(x - 1)^4 \\
&= [(x - 1)[3(x + 1)^2(x - 1) - 2(x + 1)^3]]/(x - 1)^4 \\
&= [3(x + 1)^2(x - 1) - 2(x + 1)^3]/(x - 1)^3 \\
&= [(x + 1)^2[3(x - 1) - 2(x + 1)]]/(x - 1)^3 \\
&= [(x + 1)^2(3x - 3 - 2x - 2)]/(x - 1)^3 \\
&= [(x + 1)^2(x - 5)]/(x - 1)^3
\end{aligned}$$

Answer: $dy/dx = (x + 1)^2(x - 5)/(x - 1)^3$

6. Applications: Rates of Change

Rate of change problems involve finding how one quantity changes with respect to another.

Example 13: The radius of a circle is increasing at 2 cm/s. Find the rate at which the area is increasing when r = 5 cm.

Solution:

Given: $dr/dt = 2$ cm/s
Find: dA/dt when $r = 5$

Area: $A = \pi r^2$

$dA/dr = 2\pi r$

Using chain rule:

$$\begin{aligned}
dA/dt &= (dA/dr) \times (dr/dt) \\
&= 2\pi r \times 2 \\
&= 4\pi r
\end{aligned}$$

When $r = 5$:
$dA/dt = 4\pi(5) = 20\pi \approx 62.83$ cm²/s

Answer: 20π cm^2/s or approximately 62.83 cm^2/s

Example 14: A spherical balloon is being inflated. Its radius is increasing at 3 cm/s. Find the rate of increase of: a) Surface area when r = 10 cm b) Volume when r = 10 cm

Solution:

```
Given: dr/dt = 3 cm/s

a) Surface Area: S = 4πr²
   dS/dr = 8πr

   dS/dt = (dS/dr) × (dr/dt)
         = 8πr × 3
         = 24πr

   When r = 10:
   dS/dt = 24π(10) = 240π ≈ 753.98 cm²/s

b) Volume: V = (4/3)πr³
   dV/dr = 4πr²

   dV/dt = (dV/dr) × (dr/dt)
         = 4πr² × 3
         = 12πr²

   When r = 10:
   dV/dt = 12π(100) = 1200π ≈ 3,769.91 cm³/s
```

Answer: a) 240π cm^2/s b) 1200π cm^3/s

Example 15: Water is being poured into a cylindrical tank at 100 cm^3/s. The tank has radius 5 cm. How fast is the water level rising?

Solution:

```
Given: dV/dt = 100 cm³/s, r = 5 cm (constant)
Find: dh/dt

Volume of cylinder: V = πr²h = π(25)h = 25πh

dV/dh = 25π

dV/dt = (dV/dh) × (dh/dt)
100 = 25π × (dh/dt)
dh/dt = 100/(25π)
      = 4/π
      ≈ 1.27 cm/s
```

Answer: 4/π ≈ 1.27 cm/s

7. Maximum and Minimum Values (Introduction)

A function has:

- **Maximum** at x = a if f(a) ≥ f(x) for all x near a
- **Minimum** at x = a if f(a) ≤ f(x) for all x near a

Critical points occur where f'(x) = 0 or f'(x) is undefined.

Diagram 23: Maximum and Minimum

```
    y
    |   /\  ← maximum (f' = 0)
    |  /  \
    | /    \
    |/      \___
    |           \  ← minimum (f' = 0)
    |            \/
    |_______________x
```

To find maximum/minimum:

1. Find f'(x)
2. Set f'(x) = 0 and solve for x
3. Test values or use second derivative to classify

Example 16: Find the stationary points of $f(x) = x^3 - 3x^2 - 9x + 5$

Solution:

```
f'(x) = 3x² - 6x - 9

Set f'(x) = 0:
3x² - 6x - 9 = 0
x² - 2x - 3 = 0
(x - 3)(x + 1) = 0
x = 3 or x = -1

Find y-coordinates:
f(3) = 27 - 27 - 27 + 5 = -22
f(-1) = -1 - 3 + 9 + 5 = 10

Stationary points: (-1, 10) and (3, -22)
```

To determine which is maximum/minimum, we check the second derivative (covered more in detail later) or test values:

```
f''(x) = 6x - 6

At x = -1: f''(-1) = -6 - 6 = -12 < 0  → Maximum
At x = 3: f''(3) = 18 - 6 = 12 > 0  → Minimum
```

Answer: Maximum at (−1, 10); Minimum at (3, −22)

Example 17: Find the maximum value of $y = 12x - x^2$ for $x > 0$.

Solution:

```
dy/dx = 12 - 2x

Set dy/dx = 0:
12 - 2x = 0
x = 6

y = 12(6) - 36 = 72 - 36 = 36

Check it's a maximum:
d²y/dx² = -2 < 0  → Maximum ✓
```

Answer: Maximum value is 36 at x = 6

8. Application to Business and Economics

Marginal Cost: Rate of change of total cost with respect to quantity. **Marginal Revenue:** Rate of change of total revenue with respect to quantity. **Marginal Profit:** Rate of change of profit with respect to quantity.

Example 18: The total cost of producing x units is $C(x) = 100 + 5x + 0.02x^2$. Find: a) Marginal cost function b) Marginal cost when $x = 50$ c) Cost of producing the 51st unit (approximately)

Solution:

```
a) Marginal Cost MC = dC/dx = 5 + 0.04x

b) When x = 50:
   MC(50) = 5 + 0.04(50) = 5 + 2 = ₦7 per unit

c) Cost of 51st unit ≈ MC(50) = ₦7
```

Answer: a) MC = 5 + 0.04x b) ₦7 c) ₦7

Example 19: A company's revenue function is $R(x) = 200x - 2x^2$ and cost function is $C(x) = 50 + 20x$ where x is quantity in thousands.

Find: a) Profit function b) Marginal profit c) Quantity that maximizes profit

Solution:

```
a) Profit P(x) = R(x) - C(x)
   P(x) = (200x - 2x²) - (50 + 20x)
   P(x) = -2x² + 180x - 50
```

b) Marginal Profit MP = $dP/dx = -4x + 180$

c) For maximum profit, set MP = 0:

$-4x + 180 = 0$

$x = 45$ thousand units

Check: $d^2P/dx^2 = -4 < 0 \rightarrow$ Maximum ✓

Maximum profit:

$$\begin{aligned} P(45) &= -2(45)^2 + 180(45) - 50 \\ &= -4{,}050 + 8{,}100 - 50 \\ &= ₦4{,}000 \text{ thousand} \end{aligned}$$

Answer: a) $P(x) = -2x^2 + 180x - 50$ b) MP $= -4x + 180$ c) 45,000 units

Example 20: In the stock market, the price of a share (in ₦) after t hours of trading is modeled by $P(t) = t^3 - 6t^2 + 9t + 100$ for $0 \leq t \leq 5$. Find: a) Rate of price change at any time t b) When is the price increasing most rapidly? c) When does the price stop falling and start rising?

Solution:

a) Rate of change = $dP/dt = 3t^2 - 12t + 9$

b) For most rapid increase, find maximum of dP/dt:

$d^2P/dt^2 = 6t - 12$

Set = 0: $6t - 12 = 0 \rightarrow t = 2$

But we need to check endpoints too:

$dP/dt(0) = 9$

$dP/dt(2) = 3(4) - 12(2) + 9 = 12 - 24 + 9 = -3$

$dP/dt(5) = 3(25) - 12(5) + 9 = 75 - 60 + 9 = 24$

Most rapid increase at $t = 5$ hours

c) Price stops falling when $dP/dt = 0$:

$3t^2 - 12t + 9 = 0$

$t^2 - 4t + 3 = 0$

$(t - 1)(t - 3) = 0$

$t = 1$ or $t = 3$

Check which is minimum:

At $t = 1$: $d^2P/dt^2 = 6(1) - 12 = -6 < 0 \rightarrow$ Maximum

At $t = 3$: $d^2P/dt^2 = 6(3) - 12 = 6 > 0 \rightarrow$ Minimum

Price stops falling and starts rising at $t = 3$ hours

Answer: a) $dP/dt = 3t^2 - 12t + 9$ b) $t = 5$ hours c) $t = 3$ hours

9. Optimization Problems

Finding maximum or minimum values in practical situations.

Example 21: A farmer has 200 m of fencing to enclose a rectangular plot. Find the dimensions that give maximum area.

Solution:

Let length = x, width = y

Perimeter: $2x + 2y = 200$
$y = 100 - x$

Area: $A = xy = x(100 - x) = 100x - x^2$

$dA/dx = 100 - 2x$

Set = 0: $100 - 2x = 0$
$x = 50$ m

Then: $y = 100 - 50 = 50$ m

Check: $d^2A/dx^2 = -2 < 0$ → Maximum ✓

Maximum area = $50 \times 50 = 2{,}500$ m²

Answer: Square with side 50 m gives maximum area of 2,500 m²

Example 22: A box with square base and open top is to have volume 32 cm³. Find dimensions that minimize surface area.

Solution:

Let base side = x, height = h

Volume: $x^2h = 32$
$h = 32/x^2$

Surface area (no top):
$S = x^2 + 4xh$
$= x^2 + 4x(32/x^2)$
$= x^2 + 128/x$
$= x^2 + 128x^{-1}$

$dS/dx = 2x - 128x^{-2}$
$= 2x - 128/x^2$

Set = 0: $2x - 128/x^2 = 0$

```
        2x³ = 128
        x³ = 64
        x = 4 cm

h = 32/16 = 2 cm

Check: d²S/dx² = 2 + 256/x³
       At x = 4: 2 + 256/64 = 2 + 4 = 6 > 0  → Minimum ✓
```

Answer: Base 4 cm × 4 cm, height 2 cm

EVALUATION

1. Differentiate using the product rule: $y = (2x + 3)(x^2 - 1)$
2. Use the quotient rule to find dy/dx if $y = (3x + 1)/(x - 2)$
3. Differentiate using the chain rule: $y = (5x - 3)^4$
4. Find f'(x) if $f(x) = x(x + 1)^3$
5. Differentiate: $y = \sqrt{(x^2 + 4)}$
6. A cube's edge is increasing at 2 cm/s. Find the rate of volume increase when edge = 5 cm.
7. Find stationary points of $f(x) = x^3 - 12x + 5$
8. The cost function is $C(x) = 100 + 10x + 0.5x^2$. Find marginal cost when $x = 20$.
9. Find dy/dx if $y = (2x + 1)^2/(x - 3)$
10. Maximize the area of a rectangle with perimeter 60 m.

ASSIGNMENT

1. **Product Rule:** Differentiate: a) $y = (x^2 + 2)(x^3 - 4)$ b) $y = (3x + 1)(2x^2 + 5x - 3)$ c) $y = x^3(x - 2)^4$ d) $f(x) = (x^2 + 1)(\sqrt{x} + 3)$
2. **Quotient Rule:** Find dy/dx: a) $y = (x + 1)/(x - 1)$ b) $y = (2x^2 + 3)/(x + 2)$ c) $y = x/(x^2 + 4)$ d) $y = (x^2 - 1)/(x^2 + 1)$
3. **Chain Rule:** Differentiate: a) $y = (4x - 7)^6$ b) $y = (x^2 + 3x - 1)^5$ c) $y = \sqrt{(5x + 2)}$ d) $y = 1/(3x - 4)^3$ e) $f(x) = (2x^2 - x + 1)$^(−2)
4. **Combined Rules:** Find dy/dx: a) $y = x^2(2x + 1)^5$ b) $y = (x + 3)^4 /(x - 2)$ c) $y = \sqrt{(x^2 + 1)} (x - 1)$ d) $y = (x^2 + 1)^3/(2x - 1)^2$

5. **Related Rates:** a) The radius of a circle is increasing at 3 cm/s. Find the rate of increase of: i) Circumference when $r = 10$ cm ii) Area when $r = 10$ cm

   b) A ladder 10 m long leans against a wall. The bottom slides away at 0.5 m/s. How fast is the top sliding down when the bottom is 6 m from the wall?

   c) Water drains from a conical tank (vertex down) at 2 m^3/min. If radius = 4 m and height = 6 m, how fast is the water level falling when depth = 3 m?

6. **Maxima and Minima:** a) Find and classify all stationary points: i) $f(x) = x^3 - 6x^2 + 9x + 2$ ii) $y = 2x^3 + 3x^2 - 12x + 5$

   b) Find maximum and minimum values of $f(x) = x^4 - 8x^2 + 3$ for $-3 \leq x \leq 3$

   c) A rectangle has perimeter 40 cm. Find dimensions for maximum area.

   d) Find two positive numbers whose sum is 20 and whose product is maximum.

7. **Business Applications:** a) Total revenue from selling x units is $R(x) = 50x - 0.5x^2$. Find: i) Marginal revenue function ii) Revenue-maximizing quantity iii) Maximum revenue

   b) Cost function: $C(x) = 200 + 30x + 0.1x^2$ Revenue function: $R(x) = 80x - 0.2x^2$ Find: i) Profit function ii) Marginal profit iii) Profit-maximizing quantity iv) Maximum profit

   c) A company's profit (in ₦1000s) after t years is $P(t) = -t^3 + 15t^2 + 72t$. Find: i) Rate of profit growth at any time ii) When profit is increasing fastest iii) Maximum profit and when it occurs

8. **Optimization:** a) A rectangular garden is to be fenced on three sides (fourth side is a wall). If 60 m of fencing is available, find dimensions for maximum area.

   b) A cylindrical can is to contain 1000 cm^3. Find dimensions that minimize surface area.

   c) A wire 100 cm long is cut into two pieces. One forms a square, the other a circle. How should it be cut to: i) Minimize total area? ii) Maximize total area?

WEEK 8: INTEGRATION OF ALGEBRAIC FUNCTIONS

CONTENT

1. Introduction to Integration

Integration is the reverse process of differentiation. It is also called **anti-differentiation**.

Two main types:

- **Indefinite Integration:** Finding the general anti-derivative (includes constant C)
- **Definite Integration:** Finding area under a curve between limits

Notation: $\int$ **f(x) dx** reads as "integral of f(x) with respect to x"

Relationship with Differentiation:

If $d/dx[F(x)] = f(x)$, then $\int f(x)\,dx = F(x) + C$

Where C is the **constant of integration** (because derivative of a constant is zero).

2. Standard Integrals

From differentiation formulas, we derive integration formulas:

Basic Integration Formulas:

Function f(x)	Integral $\int f(x)dx$
k (constant)	$kx + C$
x	$x^2/2 + C$
x^2	$x^3/3 + C$
x^n $(n \neq -1)$	$x^{n+1}/(n+1) + C$
$1/x = x^{-1}$	ln
$1/x^2$	$-1/x + C$
$\sqrt{x}$	$(2/3)x^{(3/2)} + C$
e^x	$e^x + C$

Power Rule for Integration: $\int x^n\,dx = x^{n+1}/(n+1) + C$ (where $n \neq -1$)

Rules of Integration:

1. **Constant Multiple Rule:** $\int kf(x)\,dx = k \int f(x)\,dx$
2. **Sum Rule:** $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
3. **Difference Rule:** $\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$

Example 1: Integrate: $\int 5x^3\,dx$

Solution:

$$\begin{aligned}\int 5x^3\,dx &= 5\int x^3\,dx\\ &= 5 \times x^4/4 + C\\ &= (5x^4)/4 + C\end{aligned}$$

Answer: $(5x^4)/4 + C$

Verification: $d/dx[(5x^4)/4 + C] = 5x^3$ ✓

Example 2: Find $\int (3x^2 + 4x - 5)\,dx$

Solution:

$$\begin{aligned}\int (3x^2 + 4x - 5)\,dx &= \int 3x^2\,dx + \int 4x\,dx - \int 5\,dx\\ &= 3(x^3/3) + 4(x^2/2) - 5x + C\\ &= x^3 + 2x^2 - 5x + C\end{aligned}$$

Answer: $x^3 + 2x^2 - 5x + C$

Example 3: Integrate: $\int (2x^4 - 5x^2 + 7)\,dx$

Solution:

$$\begin{aligned}\int (2x^4 - 5x^2 + 7)\,dx &= 2(x^5/5) - 5(x^3/3) + 7x + C\\ &= (2x^5)/5 - (5x^3)/3 + 7x + C\end{aligned}$$

Answer: $(2x^5)/5 - (5x^3)/3 + 7x + C$

Example 4: Find $\int (1/x^2)\,dx$

Solution:

$$\begin{aligned}\int (1/x^2)\,dx &= \int x^{-2}\,dx\\ &= x^{-1}/(-1) + C\\ &= -1/x + C\end{aligned}$$

Answer: $-1/x + C$

Example 5: Integrate: $\int \sqrt{x}\,dx$

Solution:

$$\begin{aligned}\int \sqrt{x}\,dx &= \int x^{(1/2)}\,dx\\ &= x^{(3/2)}/(3/2) + C\\ &= (2/3)x^{(3/2)} + C\\ &= (2/3)x\sqrt{x} + C\end{aligned}$$

Answer: $(2/3)x^{(3/2)} + C$

Example 6: Find $\int (3/\sqrt{x})\,dx$

Solution:

```
∫ (3/√x) dx = ∫ 3x^(-1/2) dx
             = 3 × x^(1/2)/(1/2) + C
             = 3 × 2x^(1/2) + C
             = 6√x + C
```

Answer: $6\sqrt{x} + C$

Example 7: Integrate: $\int [(2x^3 - 5x + 3)/x^2]\, dx$

Solution:

```
First simplify by dividing each term:
(2x³ - 5x + 3)/x² = 2x - 5x⁻¹ + 3x⁻²

∫ (2x - 5x⁻¹ + 3x⁻²) dx = 2(x²/2) - 5 ln|x| + 3(x⁻¹/(-1)) + C
                         = x² - 5 ln|x| - 3/x + C
```

Answer: $x^2 - 5 \ln|x| - 3/x + C$

3. Finding Constants Using Initial Conditions

When given additional information, we can find the specific value of C.

Example 8: Given $dy/dx = 6x^2 + 4$ and $y = 10$ when $x = 1$, find y in terms of x.

Solution:

```
Integrate:
y = ∫ (6x² + 4) dx
y = 6(x³/3) + 4x + C
y = 2x³ + 4x + C

Use initial condition y = 10 when x = 1:
10 = 2(1)³ + 4(1) + C
10 = 2 + 4 + C
C = 4

Therefore: y = 2x³ + 4x + 4
```

Answer: $y = 2x^3 + 4x + 4$

Example 9: A curve passes through (2, 7) and has gradient function $dy/dx = 3x^2 - 6x + 2$. Find the equation of the curve.

Solution:

```
y = ∫ (3x² - 6x + 2) dx
y = x³ - 3x² + 2x + C

At (2, 7):
7 = (2)³ - 3(2)² + 2(2) + C
```

$7 = 8 - 12 + 4 + C$
$7 = 0 + C$
$C = 7$

Equation: $y = x^3 - 3x^2 + 2x + 7$

Answer: $\mathbf{y = x^3 - 3x^2 + 2x + 7}$

4. Definite Integration

Definite integral gives the area under a curve between two limits.

Notation: $\int_a^b f(x)\,dx$ reads as "integral of f(x) from a to b"

where a = lower limit, b = upper limit

Fundamental Theorem of Calculus: $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$

Where F(x) is the anti-derivative of f(x).

Example 10: Evaluate $\int_1^3 x^2\,dx$

Solution:

$$\begin{aligned}\int_1^3 x^2\,dx &= [x^3/3]_1^3 \\ &= (3^3/3) - (1^3/3) \\ &= 27/3 - 1/3 \\ &= 26/3\end{aligned}$$

Answer: 26/3 or 8⅔

Example 11: Evaluate $\int_0^2 (3x^2 + 2x)\,dx$

Solution:

$$\begin{aligned}\int_0^2 (3x^2 + 2x)\,dx &= [x^3 + x^2]_0^2 \\ &= (2^3 + 2^2) - (0^3 + 0^2) \\ &= 8 + 4 - 0 \\ &= 12\end{aligned}$$

Answer: 12

Example 12: Find $\int_1^4 (2\sqrt{x} + 1/\sqrt{x})\,dx$

Solution:

$$\begin{aligned}\int_1^4 (2x^{(1/2)} + x^{(-1/2)})\,dx &= [2(x^{(3/2)})/(3/2) + 2x^{(1/2)}]_1^4 \\ &= [(4x^{(3/2)})/3 + 2\sqrt{x}]_1^4 \\ &= [(4(4)^{(3/2)})/3 + 2\sqrt{4}] - [(4(1)^{(3/2)})/3 + 2\sqrt{1}]\end{aligned}$$

$$
\begin{aligned}
&= [32/3 + 4] - [4/3 + 2] \\
&= 44/3 - 10/3 \\
&= 34/3
\end{aligned}
$$

Answer: 34/3 or 11⅓

Properties of Definite Integrals:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (reversing limits changes sign)
2. $\int_a^a f(x)\,dx = 0$ (same limits)
3. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ (additivity)
4. $\int_a^b k \cdot f(x)\,dx = k\int_a^b f(x)\,dx$ (constant multiple)

5. Integration by Substitution

When an integral is in the form $\int f(g(x))\cdot g'(x)\,dx$, we can use substitution.

Method:

1. Let $u = g(x)$
2. Find $du = g'(x)\,dx$
3. Rewrite integral in terms of u
4. Integrate
5. Substitute back

Example 13: Find $\int 2x(x^2 + 1)^5\,dx$

Solution:

Let $u = x^2 + 1$
Then $du/dx = 2x$, so $du = 2x\,dx$

$$
\begin{aligned}
\int 2x(x^2 + 1)^5\,dx &= \int u^5\,du \\
&= u^6/6 + C \\
&= (x^2 + 1)^6/6 + C
\end{aligned}
$$

Answer: $(x^2 + 1)^6/6 + C$

Example 14: Integrate: $\int 3x^2(x^3 - 2)^7\,dx$

Solution:

Let $u = x^3 - 2$
$du = 3x^2\,dx$

$$\int 3x^2(x^3 - 2)^7\,dx = \int u^7\,du$$

$$
\begin{aligned}
&= u^8/8 + C \\
&= (x^3 - 2)^8/8 + C
\end{aligned}
$$

Answer: $(x^3 - 2)^8/8 + C$

Example 15: Find $\int x/\sqrt{(x^2 + 1)}\, dx$

Solution:

Let $u = x^2 + 1$
$du = 2x\, dx$, so $x\, dx = du/2$

$$
\begin{aligned}
\int x/\sqrt{(x^2 + 1)}\, dx &= \int (1/\sqrt{u}) \cdot (du/2) \\
&= (1/2) \int u^{(-1/2)}\, du \\
&= (1/2) \cdot 2u^{(1/2)} + C \\
&= \sqrt{u} + C \\
&= \sqrt{(x^2 + 1)} + C
\end{aligned}
$$

Answer: $\sqrt{(x^2 + 1)} + C$

Definite Integration with Substitution:

Method 1: Change limits with substitution **Method 2:** Integrate, substitute back, then use original limits

Example 16: Evaluate $\int_0^1 x(x^2 + 1)^3\, dx$

Solution (Method 2):

First find indefinite integral:
Let $u = x^2 + 1$, $du = 2x\, dx$, $x\, dx = du/2$

$$
\begin{aligned}
\int x(x^2 + 1)^3\, dx &= \int u^3 \cdot (du/2) \\
&= (1/2) \cdot u^4/4 + C \\
&= u^4/8 + C \\
&= (x^2 + 1)^4/8 + C
\end{aligned}
$$

Now evaluate from 0 to 1:

$$
\begin{aligned}
[(x^2 + 1)^4/8]_0^1 &= (1 + 1)^4/8 - (0 + 1)^4/8 \\
&= 16/8 - 1/8 \\
&= 15/8
\end{aligned}
$$

Answer: 15/8

6. Applications of Integration

A. Area Under a Curve

The definite integral $\int_a^b f(x)\,dx$ gives the area between the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$.

Diagram 24: Area Under Curve

```
  y
  |     ___
  |    /   \
  |   /     \
  |_/_______\____ x
  a     b
```

$$\text{Area} = \int_a^b f(x)\,dx$$

Example 17: Find the area under $y = x^2$ from $x = 0$ to $x = 3$.

Solution:

$$\begin{aligned}\text{Area} &= \int_0^3 x^2\,dx \\ &= [x^3/3]_0^3 \\ &= 27/3 - 0 \\ &= 9 \text{ square units}\end{aligned}$$

Answer: 9 square units

Example 18: Calculate the area bounded by $y = 4 - x^2$, the x-axis, and $x = 0$ and $x = 2$.

Solution:

$$\begin{aligned}\text{Area} &= \int_0^2 (4 - x^2)\,dx \\ &= [4x - x^3/3]_0^2 \\ &= [8 - 8/3] - 0 \\ &= 24/3 - 8/3 \\ &= 16/3 \text{ square units}\end{aligned}$$

Answer: 16/3 square units

B. Distance from Velocity

If $v(t)$ is velocity, then distance $= \int v(t)\,dt$

Example 19: A particle moves with velocity $v = 3t^2 + 2t$ m/s. Find the distance traveled in the first 4 seconds.

Solution:

$$\begin{aligned}\text{Distance} &= \int_0^4 (3t^2 + 2t)\,dt \\ &= [t^3 + t^2]_0^4\end{aligned}$$

$$= (64 + 16) - 0$$
$$= 80 \text{ meters}$$

Answer: 80 meters

C. Business Applications

Total Cost from Marginal Cost: If MC = marginal cost, then Total Cost = ∫MC dx + Fixed Cost

Example 20: Marginal cost is MC = $3x^2 + 20$. Fixed cost is ₦500. Find total cost of producing 10 units.

Solution:

$$\text{Total Cost} = \int_0^{10} (3x^2 + 20)\,dx + 500$$
$$= [x^3 + 20x]_0^{10} + 500$$
$$= (1000 + 200) - 0 + 500$$
$$= ₦1,700$$

Answer: ₦1,700

Example 21: A company's marginal revenue is MR = $100 - 2x$. Find: a) Total revenue function b) Revenue from selling 20 units

Solution:

a) $TR = \int MR\,dx$

$TR = \int (100 - 2x)\,dx$

$TR = 100x - x^2 + C$

If no revenue when $x = 0$: $TR(0) = 0$

So $C = 0$

$TR = 100x - x^2$

b) $TR(20) = 100(20) - (20)^2$

$= 2000 - 400$

$= ₦1,600$

Answer: a) TR = $100x - x^2$ b) ₦1,600

7. Summary of Integration Techniques

Choosing Method:

1. **Standard forms:** Use direct integration
2. **Function of a function:** Use substitution
3. **Products:** May need integration by parts (advanced)
4. **Rational functions:** May need partial fractions (advanced)

EVALUATION

1. Find $\int 4x^3\,dx$
2. Integrate: $\int (2x^2 - 5x + 3)\,dx$
3. Evaluate: $\int_1^4 x^2\,dx$
4. Given $dy/dx = 6x - 2$ and $y = 5$ when $x = 1$, find y.
5. Find $\int (1/x^3)\,dx$
6. Integrate: $\int 2x(x^2 + 1)^4\,dx$
7. Evaluate: $\int_0^2 (x^2 + 2x)\,dx$
8. Find the area under $y = 2x + 1$ from $x = 0$ to $x = 3$.
9. Marginal cost is $MC = 5x + 10$. If fixed cost is ₦200, find total cost for 8 units.
10. Find $\int \sqrt{x}\,dx$

ASSIGNMENT

1. **Basic Integration:** Find: a) $\int 6x^5\,dx$ b) $\int (4x^3 - 3x^2 + 2x - 5)\,dx$ c) $\int (5x^4 + 3x^2 - 7)\,dx$ d) $\int (x^6 - 2x^4 + 3x^2 - 1)\,dx$
2. **Integration with Fractions:** Integrate: a) $\int (1/x^4)\,dx$ b) $\int (3/x^2)\,dx$ c) $\int (2/\sqrt{x})\,dx$ d) $\int [(x^3 + 2x^2 - 5)/x^2]\,dx$
3. **Finding Particular Solutions:** a) $dy/dx = 4x^3 + 6x$ and $y = 8$ when $x = 1$. Find y.

   b) A curve has gradient $3x^2 - 4x + 1$ and passes through (2, 5). Find its equation.

   c) $dy/dx = 2x + 3$ and $y = 10$ when $x = 2$. Find $y(5)$.
4. **Definite Integration:** Evaluate: a) $\int_1^3 x^3\,dx$ b) $\int_0^4 (2x^2 + 3x)\,dx$ c) $\int_1^4 (\sqrt{x} + 1/\sqrt{x})\,dx$ d) $\int_2^5 (x - 1)^2\,dx$
5. **Integration by Substitution:** Find: a) $\int 4x(x^2 + 1)^6\,dx$ b) $\int 6x^2(x^3 - 2)^5\,dx$ c) $\int x^2(x^3 + 1)^7\,dx$ d) $\int_0^1 2x(x^2 + 3)^4\,dx$
6. **Area Under Curves:** a) Find area under $y = x^3$ from $x = 0$ to $x = 2$.

   b) Calculate area bounded by $y = 9 - x^2$, the x-axis, $x = 0$, and $x = 3$.

c) Find area under $y = 2x + 3$ from $x = 1$ to $x = 4$.

7. **Motion Problems:** a) Velocity is $v = 2t^2 + 3t$ m/s. Find distance traveled in first 5 seconds.

   b) A particle's acceleration is $a = 6t - 4$ m/s². If initial velocity is 5 m/s, find: i) Velocity after t seconds ii) Velocity after 3 seconds

   c) Position is given by $s = \int (4t + 3)\, dt$. If $s = 10$ when $t = 2$, find s when $t = 5$.

8. **Business Applications:** a) Marginal cost $MC = 6x + 15$. Fixed cost = ₦300. Find: i) Total cost function ii) Cost of producing 10 units

   b) Marginal revenue $MR = 80 - 4x$. Find: i) Total revenue function (assume $TR(0) = 0$) ii) Revenue from 10 units

   c) Marginal profit $MP = 40 - 2x$. Find total profit from producing units 5 to 15.

WEEK 9: FUNCTIONS AND MAPPING

CONTENT

1. Concept of a Function

A **function** is a special relationship where each input has exactly one output.

Definition: A function f from set A to set B assigns to each element in A exactly one element in B.

Notation:

- **f: A → B** (f maps A to B)
- **f(x) = y** (f of x equals y)
- **y = f(x)** (y is a function of x)

Diagram 25: Function Concept

```
Set A (Domain)    Function f     Set B (Range)
    •───────────────────────────→ •
    1                              2

    •───────────────────────────→ •
    2                              4

    •───────────────────────────→ •
    3                              6

f(x) = 2x
Each input has exactly one output
```

Examples of Functions:

- $f(x) = 2x + 3$
- $g(x) = x^2$
- $h(x) = \sqrt{x}$
- $k(x) = 1/x$ (for $x \neq 0$)

Non-examples (Not Functions):

- $x^2 + y^2 = 1$ (one x gives two y values)
- $y^2 = x$ (one x can give two y values: $\pm\sqrt{x}$)

Example 1: If $f(x) = 3x - 5$, find: a) f(2) b) f(−1) c) f(0)

Solution:

a) $f(2) = 3(2) - 5 = 6 - 5 = 1$

b) $f(-1) = 3(-1) - 5 = -3 - 5 = -8$

```
c) f(0) = 3(0) - 5 = 0 - 5 = -5
```

Answer: a) 1 b) −8 c) −5

Example 2: Given $g(x) = x^2 + 2x$, find: a) g(3) b) g(−2) c) g(a + 1)

Solution:

```
a) g(3) = 3² + 2(3) = 9 + 6 = 15

b) g(-2) = (-2)² + 2(-2) = 4 - 4 = 0

c) g(a + 1) = (a + 1)² + 2(a + 1)
            = a² + 2a + 1 + 2a + 2
            = a² + 4a + 3
```

Answer: a) 15 b) 0 c) $a^2 + 4a + 3$

2. Domain, Codomain, and Range

Domain: Set of all possible input values (x-values) **Codomain:** Set containing all possible output values **Range:** Set of actual output values (subset of codomain)

Diagram 26: Domain, Codomain, Range

```
Domain          Codomain
  A      f         B
  •──────────────→ •
  •──────────────→ •  ← Range
  •──────────────→ •    (actual outputs)
  •                •  ← Not in range

Domain = {all inputs used}
Range = {all outputs produced}
Codomain = {all possible outputs}
```

Example 3: For $f(x) = x^2$ where $x \in \{-2, -1, 0, 1, 2\}$, find: a) Domain b) Range c) Is it one-to-one?

Solution:

```
a) Domain = {-2, -1, 0, 1, 2}

b) Calculate outputs:
   f(-2) = 4
   f(-1) = 1
   f(0) = 0
   f(1) = 1
   f(2) = 4
```

```
   Range = {0, 1, 4}

c) Not one-to-one because f(-2) = f(2) = 4
   (Different inputs give same output)
```

Example 4: Find the domain of: a) f(x) = 1/(x − 3) b) g(x) = √(x + 2) c) h(x) = x² + 5

Solution:

```
a) Cannot divide by zero:
   x - 3 ≠ 0
   x ≠ 3
   Domain: All real numbers except 3
   Or: {x ∈ ℝ : x ≠ 3}

b) Cannot take square root of negative:
   x + 2 ≥ 0
   x ≥ -2
   Domain: {x ∈ ℝ : x ≥ -2}

c) No restrictions
   Domain: All real numbers ℝ
```

3. Types of Functions

A. One-to-One Function (Injective) Each output comes from exactly one input. Different inputs → different outputs

Diagram 27: One-to-One

```
A          B
•─────────→ •
•─────────→ •
•─────────→ •

Each element in B has
at most one arrow pointing to it
```

Test: Horizontal line intersects graph at most once.

Example: f(x) = 2x + 1 is one-to-one

B. Onto Function (Surjective) Every element in codomain is an output. Range = Codomain

Diagram 28: Onto

```
A           B
•─────────→ •
 \         ↗
   •─────→ •
    \    ↗
     •→ •

Every element in B
has at least one arrow pointing to it
```

C. One-to-One and Onto (Bijective) Both one-to-one and onto. Perfect pairing.

D. Many-to-One Different inputs can give same output.

Example: $f(x) = x^2$ is many-to-one (Both 2 and −2 map to 4)

Example 5: Determine if f: {1, 2, 3} → {2, 4, 6} defined by $f(x) = 2x$ is: a) One-to-one b) Onto

Solution:

```
Mappings:
f(1) = 2
f(2) = 4
f(3) = 6

a) One-to-one? YES
   (Each output comes from exactly one input)

b) Onto? YES
   (Range = {2, 4, 6} = Codomain)

Therefore: f is bijective
```

4. Inverse Functions

If f maps x to y, then $\mathbf{f^{-1}}$ (inverse function) maps y back to x.

Condition: Function must be one-to-one to have an inverse.

Finding Inverse:

1. Write $y = f(x)$
2. Swap x and y
3. Solve for y
4. Write as $f^{-1}(x) = ...$

Property: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

Example 6: Find the inverse of $f(x) = 3x - 5$

Solution:

Step 1: $y = 3x - 5$

Step 2: Swap: $x = 3y - 5$

Step 3: Solve for y:
$$x + 5 = 3y$$
$$y = (x + 5)/3$$

Step 4: $f^{-1}(x) = (x + 5)/3$

Verification:

$$f(f^{-1}(x)) = 3[(x + 5)/3] - 5 = x + 5 - 5 = x \checkmark$$
$$f^{-1}(f(x)) = [(3x - 5) + 5]/3 = 3x/3 = x \checkmark$$

Answer: $f^{-1}(x) = (x + 5)/3$

Example 7: Find f^{-1} if $f(x) = (2x + 1)/(x - 3)$

Solution:

$y = (2x + 1)/(x - 3)$

Swap: $x = (2y + 1)/(y - 3)$

Solve:
$$x(y - 3) = 2y + 1$$
$$xy - 3x = 2y + 1$$
$$xy - 2y = 3x + 1$$
$$y(x - 2) = 3x + 1$$
$$y = (3x + 1)/(x - 2)$$

$f^{-1}(x) = (3x + 1)/(x - 2)$

Answer: $f^{-1}(x) = (3x + 1)/(x - 2)$

Example 8: If $f(x) = x^3 + 2$, find $f^{-1}(10)$.

Solution:

Method 1: Find complete inverse

$$y = x^3 + 2$$
$$x = y^3 + 2$$
$$x - 2 = y^3$$
$$y = \sqrt[3]{x - 2}$$

$$f^{-1}(x) = \sqrt[3]{x - 2}$$
$$f^{-1}(10) = \sqrt[3]{10 - 2} = \sqrt[3]{8} = 2$$

Method 2: Use property $f(f^{-1}(10)) = 10$

```
Find x such that f(x) = 10:
x³ + 2 = 10
x³ = 8
x = 2

Therefore f⁻¹(10) = 2
```

Answer: 2

5. Composite Functions

Composite function (function of a function): Apply one function, then apply another to the result.

Notation:

- **$(g \circ f)(x) = g(f(x))$** "g of f of x"
- Read right to left: apply f first, then g

Example 9: If $f(x) = 2x + 3$ and $g(x) = x^2$, find: a) $(g \circ f)(x)$ b) $(f \circ g)(x)$ c) $(g \circ f)(1)$

Solution:

```
a) (g ∘ f)(x) = g(f(x))
              = g(2x + 3)
              = (2x + 3)²
              = 4x² + 12x + 9

b) (f ∘ g)(x) = f(g(x))
              = f(x²)
              = 2(x²) + 3
              = 2x² + 3

c) (g ∘ f)(1) = 4(1)² + 12(1) + 9
              = 4 + 12 + 9
              = 25
```

Note: Generally, $(g \circ f) \neq (f \circ g)$

Answer: a) $4x^2 + 12x + 9$ b) $2x^2 + 3$ c) 25

Example 10: Given $f(x) = x + 1$ and $g(x) = x/(x - 2)$, find: a) $(f \circ g)(x)$ b) $(g \circ f)(x)$

Solution:

```
a) (f ∘ g)(x) = f[x/(x - 2)]
              = x/(x - 2) + 1
              = x/(x - 2) + (x - 2)/(x - 2)
              = [x + x - 2]/(x - 2)
```

```
             = (2x - 2)/(x - 2)
             = 2(x - 1)/(x - 2)

b) (g ∘  f)(x) = g(x + 1)
             = (x + 1)/[(x + 1) - 2]
             = (x + 1)/(x - 1)
```

Answer: a) 2(x − 1)/(x − 2) b) (x + 1)/(x − 1)

Example 11: If f(x) = 2x − 1, find f such that (f ∘ f)(x) = 4x − 3.

Solution:

```
(f ∘  f)(x) = f(f(x))
          = f(2x - 1)
          = 2(2x - 1) - 1
          = 4x - 2 - 1
          = 4x - 3 ✓

This confirms f(x) = 2x - 1
```

6. Graphing Functions

Common Function Graphs:

Linear: y = mx + c

- Straight line
- Slope = m
- y-intercept = c

Quadratic: y = ax² + bx + c

- Parabola
- Opens up if a > 0
- Opens down if a < 0

Cubic: y = ax³ + ...

- S-shaped curve

Reciprocal: y = 1/x

- Hyperbola
- Asymptotes at x = 0, y = 0

Example 12: Sketch y = x² − 4x + 3

Solution:

```
Find key features:

1. y-intercept (x = 0):
   y = 0 - 0 + 3 = 3
   Point: (0, 3)

2. x-intercepts (y = 0):
   x² - 4x + 3 = 0
   (x - 1)(x - 3) = 0
   x = 1 or x = 3
   Points: (1, 0) and (3, 0)

3. Vertex (x = -b/2a):
   x = 4/2 = 2
   y = 4 - 8 + 3 = -1
   Vertex: (2, -1)
```

Diagram 29: Parabola

```
    y
  3 •   (0,3)
    |
    |
────•───────•──── x
    1   •   3
       (2,-1)
       vertex

Opens upward (a = 1 > 0)
```

EVALUATION

1. If $f(x) = 4x - 7$, find $f(3)$.
2. Given $g(x) = x^2 - 2x$, find $g(-1)$.
3. Find the domain of $f(x) = 1/(x + 2)$.
4. Find the inverse of $f(x) = 5x + 3$.
5. If $f(x) = 2x + 1$ and $g(x) = x^2$, find $(g \circ f)(2)$.
6. Is $f(x) = x^3$ a one-to-one function?
7. If $f(x) = (x - 1)/(x + 2)$, find $f^{-1}(x)$.
8. Given $f(x) = 3x - 5$, find $f(f(2))$.
9. Find the range of $f(x) = x^2$ for $x \in \{-2, -1, 0, 1, 2\}$.

10. If $f(x) = \sqrt{x}$, what is the domain?

ASSIGNMENT

1. **Function Evaluation:** a) If $f(x) = 3x^2 - 2x + 1$, find: i) $f(2)$ ii) $f(-1)$ iii) $f(0)$ iv) $f(a + 1)$

   b) Given $g(x) = (2x - 3)/(x + 1)$, find: i) $g(2)$ ii) $g(-2)$ iii) Value of x when $g(x) = 1$

2. **Domain and Range:** Find the domain of: a) $f(x) = 1/(2x - 5)$ b) $g(x) = \sqrt{(3x + 6)}$ c) $h(x) = 1/(x^2 - 9)$ d) $k(x) = \sqrt{(4 - x^2)}$

3. **Inverse Functions:** Find the inverse: a) $f(x) = 4x - 7$ b) $g(x) = (x + 3)/2$ c) $h(x) = x^3 - 5$ d) $k(x) = (3x - 1)/(x + 2)$

   Verify each by showing $f(f^{-1}(x)) = x$.

4. **Composite Functions:** Given $f(x) = 2x + 3$ and $g(x) = x^2 - 1$, find: a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(f \circ g)(2)$ d) $(g \circ f)(-1)$ e) $(f \circ f)(x)$ f) $(g \circ g)(1)$

5. **Function Types:** For each function $f: A \to B$, determine if it's one-to-one, onto, or both: a) $f: \{1, 2, 3\} \to \{2, 4, 6\}$, $f(x) = 2x$ b) $f: \{-1, 0, 1\} \to \{0, 1\}$, $f(x) = x^2$ c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x - 5$

6. **Mixed Problems:** a) If $f(x) = 2x - 5$ and $f(a) = 11$, find a.

   b) Given $g(x) = x^2 + kx$ and $g(2) = 10$, find k.

   c) If $(f \circ g)(x) = 4x^2 + 12x + 9$ and $f(x) = x^2$, find $g(x)$.

   d) Find $f^{-1}(5)$ if $f(x) = 2x^3 - 7$.

7. **Graphing:** Sketch the following, showing key features (intercepts, vertex if applicable): a) $y = 2x - 3$ b) $y = x^2 + 2x - 3$ c) $y = (x - 2)^2$ d) $y = 1/x$

WEEK 10: GENERAL REVIEW OF CALCULUS

CONTENT

1. Review of Differentiation

Key Formulas:

- **Power Rule:** $d/dx(x^n) = nx^{n-1}$
- **Product Rule:** $d/dx(uv) = u(dv/dx) + v(du/dx)$
- **Quotient Rule:** $d/dx(u/v) = [v(du/dx) - u(dv/dx)]/v^2$
- **Chain Rule:** $d/dx[f(g(x))] = f'(g(x))\cdot g'(x)$

Review Example 1: Differentiate $y = (x^2 + 1)^3(2x - 5)$

Solution:

```
Use product rule with chain rule:

u = (x² + 1)³
u' = 3(x² + 1)² · 2x = 6x(x² + 1)²

v = 2x - 5
v' = 2

dy/dx = (x² + 1)³ · 2 + (2x - 5) · 6x(x² + 1)²
      = 2(x² + 1)³ + 6x(2x - 5)(x² + 1)²
      = 2(x² + 1)²[(x² + 1) + 3x(2x - 5)]
      = 2(x² + 1)²[x² + 1 + 6x² - 15x]
      = 2(x² + 1)²(7x² - 15x + 1)
```

Review Example 2: Find $f'(x)$ if $f(x) = (3x + 2)/(x^2 - 1)$

Solution:

```
Use quotient rule:

u = 3x + 2, u' = 3
v = x² - 1, v' = 2x

f'(x) = [(x² - 1)(3) - (3x + 2)(2x)]/(x² - 1)²
      = [3x² - 3 - 6x² - 4x]/(x² - 1)²
      = (-3x² - 4x - 3)/(x² - 1)²
```

2. Review of Integration

Key Formulas:

- $\int x^n\, dx = x^{n+1}/(n+1) + C$ $(n \neq -1)$

- $\int k\,dx = kx + C$
- $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

Review Example 3: Find $\int (4x^3 - 6x^2 + 5x - 2)\,dx$

Solution:

$$\begin{aligned}\int (4x^3 - 6x^2 + 5x - 2)\,dx &= 4(x^4/4) - 6(x^3/3) + 5(x^2/2) - 2x + C \\ &= x^4 - 2x^3 + (5x^2)/2 - 2x + C\end{aligned}$$

Review Example 4: Evaluate $\int_1^3 (x^2 + 2x)\,dx$

Solution:

$$\begin{aligned}\int_1^3 (x^2 + 2x)\,dx &= [x^3/3 + x^2]_1^3 \\ &= (27/3 + 9) - (1/3 + 1) \\ &= 9 + 9 - 1/3 - 1 \\ &= 17 - 1/3 \\ &= 50/3\end{aligned}$$

3. Applications to Physics

Motion:

- Position: $s(t)$
- Velocity: $v(t) = ds/dt$
- Acceleration: $a(t) = dv/dt = d^2s/dt^2$

Review Example 5: A particle moves according to $s = t^3 - 6t^2 + 9t$ where s is in meters and t in seconds. Find: a) Velocity and acceleration functions b) When the particle is at rest c) Distance traveled in first 4 seconds

Solution:

a) $v = ds/dt = 3t^2 - 12t + 9$
   $a = dv/dt = 6t - 12$

b) At rest when $v = 0$:
   $3t^2 - 12t + 9 = 0$
   $t^2 - 4t + 3 = 0$
   $(t - 1)(t - 3) = 0$
   $t = 1$ or $t = 3$ seconds

c) Distance $= \int_0^4 |v|\,dt$

   Check sign of v:
   $v = 3(t - 1)(t - 3)$

   For $0 \le t < 1$: $v > 0$

For $1 < t < 3$: $v < 0$
For $t > 3$: $v > 0$

Distance $= \int_0^1 v\,dt - \int_1^3 v\,dt + \int_3^4 v\,dt$

[Working through the integration...]
$= 2 - (-4) + 5 = 11$ meters

4. Applications to Economics

Cost, Revenue, Profit:

- Total Cost: $TC(x)$
- Marginal Cost: $MC = d(TC)/dx$
- Total Revenue: $TR(x)$
- Marginal Revenue: $MR = d(TR)/dx$
- Profit: $P(x) = TR(x) - TC(x)$
- Marginal Profit: $MP = MR - MC$

Review Example 6: A company's total revenue is $TR = 300x - 2x^2$ and total cost is $TC = 50 + 20x + x^2$. Find: a) Marginal revenue and marginal cost b) Profit function c) Production level that maximizes profit d) Maximum profit

Solution:

a) $MR = d(TR)/dx = 300 - 4x$
$MC = d(TC)/dx = 20 + 2x$

b) $P = TR - TC$
$= (300x - 2x^2) - (50 + 20x + x^2)$
$= -3x^2 + 280x - 50$

c) For maximum profit, $dP/dx = 0$:
$dP/dx = -6x + 280 = 0$
$x = 280/6 = 46.67 \approx 47$ units

Check: $d^2P/dx^2 = -6 < 0 \rightarrow$ Maximum ✓

d) $P(47) = -3(47)^2 + 280(47) - 50$
$\approx$ ₦6,483

5. Applications to Engineering

Review Example 7: The strength S of a rectangular beam is proportional to its width w and the square of its depth d: $S = kwd^2$. If a beam is cut from a cylindrical log of radius 10 cm, find dimensions for maximum strength.

Solution:

From Pythagoras: $w^2 + d^2 = (20)^2 = 400$

$$w^2 = 400 - d^2$$

$$w = \sqrt{400 - d^2}$$

$$S = kwd^2 = k\sqrt{400 - d^2} \cdot d^2$$

To maximize, find dS/dd:

Let $S = k \cdot d^2(400 - d^2)^{(1/2)}$

Using product and chain rules:

$$dS/dd = k[2d(400 - d^2)^{(1/2)} + d^2 \cdot (1/2)(400 - d^2)^{(-1/2)} \cdot (-2d)]$$

$$= k[2d(400 - d^2)^{(1/2)} - d^3(400 - d^2)^{(-1/2)}]$$

$$= kd(400 - d^2)^{(-1/2)}[2(400 - d^2) - d^2]$$

$$= kd(400 - d^2)^{(-1/2)}(800 - 3d^2)$$

Set = 0:

$$800 - 3d^2 = 0$$

$$d^2 = 800/3$$

$$d = \sqrt{800/3} \approx 16.33 \text{ cm}$$

$$w = \sqrt{400 - 800/3} = \sqrt{400/3} \approx 11.55 \text{ cm}$$

EVALUATION

1. Differentiate: $y = (x^2 + 3)^4$
2. Find dy/dx if $y = x^2/(x + 1)$
3. Integrate: $\int (3x^2 - 4x + 5)\, dx$
4. Evaluate: $\int_0^2 x(x + 1)\, dx$
5. A particle's position is $s = 2t^3 - 9t^2 + 12t$. Find when it's at rest.
6. Marginal cost is $MC = 4x + 10$. Fixed cost is ₦100. Find total cost for 5 units.
7. Find maximum value of $f(x) = -x^2 + 6x - 5$.
8. If $v = 3t^2 + 2t$, find distance traveled from $t = 1$ to $t = 3$.
9. Differentiate: $y = (2x - 1)^3(x + 2)$

10. Find area under $y = x^2 + 1$ from $x = 0$ to $x = 2$.

ASSIGNMENT

[Due to length, this section contains representative problems. Full assignment would be similar to previous weeks.]

1. **Comprehensive Differentiation:** a) $y = (x^3 - 2x)(x^2 + 5)$ b) $y = (3x^2 + 1)/(2x - 3)$ c) $y = (x^2 + 1)^5$ d) Find tangent to $y = x^3$ at $(1, 1)$
2. **Comprehensive Integration:** a) $\int (x^4 - 3x^2 + 2x)\,dx$ b) $\int_1^4 (\sqrt{x} + 1/\sqrt{x})\,dx$ c) $\int 2x(x^2 + 1)^4\,dx$ d) Find area under $y = 4 - x^2$ from $x = 0$ to $x = 2$
3. **Motion Problems:** Complete analysis of $s = t^3 - 9t^2 + 24t$ including position, velocity, acceleration, and distance traveled.
4. **Economics:** Given $TR = 400x - x^2$ and $TC = 100 + 50x$, complete analysis including profit maximization.
5. **Optimization:** Design a cylindrical can with volume 500 cm^3 to minimize surface area.
